THE OBJECT AND NATURE OF MATHEMATICAL SCIENCE

IN ARISTOTLE AND ST. THOMAS AQUINAS:

A COMPARISON

by

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A Dissertation submitted to the Faculty of the Graduate School, Marquette University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

Milwaukee, Wisconsin
May, 1966

PREFACE

The following dissertation attempts to set forth the nature and object of the science of mathematics according to Aristotle and St. Thomas Aquinas, and to compare the doctrines of these two men.

At the present there are very few works which compare the teaching of the two philosophers in the area of mathematics, and none of them are extensive. Many authors do make reference to one or other points of similarity and/or difference between the doctrine of the two men, but I know of no comprehensive treatment of this topic.[1] One reason for this seems to be that it is often taken for granted that the two men have identical teachings. Indeed, as we shall show throughout this paper, many contemporary historians of philosophy very clearly but apparently unknowingly attribute positions in the philosophy of mathematics of Thomas Aquinas to his Greek predecessor. In a sense the purpose of this paper is to see if such attribution is justified, or, to put it another way, to determine whether or not there is an essential difference between the ancient philosopher and his medieval disciple in the philosophy of mathematics.

One thing should be made very clear from the beginning. When we speak of the philosophy of mathematics of

either of these thinkers we mean by mathematics the area of knowledge which was so named in their respective times. In general, for both men the mathematics which existed at their time and of which they were aware comprised what we would call today arithmetic and plane and solid Euclidean geometry, as well as the various sciences which apply these to physical phenomena.[2] (In Thomas's time the algebra was also known. The question of how much awareness of it St. Thomas had will be discussed below.) This limitation must always be kept in mind in a discussion of their mathematical doctrines. No attempt will be made in this dissertation to relate their teachings to kinds of mathematics of which they were totally unaware, such as the calculus or modern set theory. I will leave to others, with more knowledge of the mathematics of our day than I, the very important task of determining the relevancy of the teaching of these two men to our modern science (or sciences) of the same name. It is hoped that the first step in such a determination has been taken in the following pages.

Acknowledgements: I wish to express my thanks to Dr. Francis Collingwood for his guidance in the direction of this dissertation, to Dr. Edward Simmons for his various efforts in its behalf, and to my other readers, Rev. Gerard Smith, S.J., Rev. Michael Murray, S.J., and Dr. Lottie Kendzierski.

FOOTNOTES

[1]Some of the more significant works which have compared various points of the philosophy of mathematics of the two men are: Thomas Greenwood, Études sur La Connaissance Mathématique (Ottawa, 1942); Armand Maurer in his Introduction to The Division and Methods of the Sciences: Questions V and VI of His Commentary on the "De Trinitate" of Boethius by St. Thomas Aquinas (Toronto, 1953); L. B. Geiger, "Abstraction et separation d'après S. Thomas," Revue Des Sciences Philosophiques et Théologiques, XXI (1947); Frère Augustin-Gabriel, "Matière intelligible et mathématique," Laval Théologique et Philosophique, XVIII (1962), nos. 1 and 2 ; Joseph Owens, "Our Knowledge of Nature," Proceedings of the American Catholic Philosophical Association, XXIX (1955). Others will be cited throughout the paper but none of these works is extensive.

[2]For a list of the branches of mathematics at Aristotle's time, see Thomas Heath, A History of Greek Mathematics (Oxford, 1960), I, pp. 10-18 and B. L. van der Waerden, Science Awakening (New York, 1964), Chapter 6. Heath lists arithmetic, geometry, mechanics, astronomy, optics, geodesy, harmonics, and logistics. All of these can be reduced to arithmetic and geometry, pure and applied to physical phenomena. (See my text pp.145-6, on how the applied sciences of mechanics, astronomy, optics and harmonics are subordinated to geometry and/or arithmetic; see also footnote 64 of Ch. IV concerning geodesy and logistics.)

For the branches of mathematics at the time of Thomas Aquinas, see F. Cajori, A History of Mathematics (New York, 1951), pp. 113-129, the chapter entitled "Europe During the Middle Ages" and A. C. Crombie, Medieval and Early Modern Science (Garden City, New York, 1959), I, 181-183. They are substantially the same as those at Aristotle's time though algebra is also present in the thirteenth century. Thomas's knowledge of the latter, or rather the lack of it, will be discussed at length in Chapter IV of Part II of this dissertation.

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PART I

THE OBJECT AND NATURE OF MATHEMATICAL SCIENCE IN ARISTOTLE

CHAPTER I

THE OBJECT OF MATHEMATICS ACCORDING TO ARISTOTLE

Mathematics studies quantities. The Stagirite makes this perfectly clear. In the following text he equates the "proper matter" of mathematics with "some kind of quantity."

> [M]athematics studies a part of its proper matter . . . e.g. lines or angles or number or some other kind of quantity. . . .[1]

Quantity is one of the Aristotelian categories, one of the highest most general classes into which things or their aspects can be placed. It is one of that list of the most general possible predicates of all things.[2] Because the categories are the ultimate classifications or genera of being (Aristotle calls them "the kinds of essential being"),[3] it is impossible strictly speaking to define any of them for to define something is to place it in a genus and to add that difference which specifically distinguishes it from other members of that genus.[4] However, we are still able to talk about each category, to describe it, and to distinguish it from the others; and in fact Aristotle's work, The Categories, does just that. In the chapter of that treatise which deals with quantity (Chapter 6), he does not, however, discuss the essential nature of quantity itself but rather treats of the various attributes of quantity. He seems there to presuppose

that his readers know at least in general what he means by quantity. It is in Chapter 13 of Book V of the Metaphysics that Aristotle is more explicit on the specific nature of quantity.

> The quantitative [he says] is that which is divisible into two or more constituent parts of which each is by nature a "one" and a "this."[5]

This statement needs explanation for it describes quantity, or rather the quantitative, in terms of potency, "that which is divisible," and we need to ask — what actual feature must a thing possess in order to have this kind of potency? In other words, what is the act which allows a thing to be divisible into two or more constituent parts, the act which would apparently be the attribute of quantity? Also, the above statement's description of parts as "one" and "this" needs clarification. We will attempt this clarification first.

To say that each of the parts is by nature a "one" and a "this" does not necessarily mean that these parts can exist as distinct individual substances — though this is sometimes the case. Certainly a solid block of wood is quantitative and it can be divided into parts which are capable of existing on their own apart from the original block. Likewise, a group of ten fish is apparently quantitative for the parts of this totality, viz. the individual fish, are capable of being divided off from the totality and of existing in their own right after such division. In each of these apparent quantities, the parts are capable of independent

existence. However, Aristotle also refers to other quantities, for example numbers, whose parts do not have this capability. (As we shall discuss in detail in the next chapter, number is not a substance. Therefore, neither can its parts be substances.)

Actually, when the Stagirite refers to the parts of the quantitative as by nature "one," he means fundamentally that they are in some way indivisible. "[I]n general these things that do not admit of division are called one insofar as they do not admit of it. . . ."[6] The term "one," he says, has many meanings, as many as there are meanings of being[7] (and being is said of accidents as well as substances), but in every case "to be one means to be indivisible."[8] Hence the parts of a divided quantity must be "ones," indivisible. To say this, that the parts must be ones, does not necessarily mean that they must be <u>completely</u> indivisible, but only that they must be indivisible in some way.[9] For example, Aristotle points out that we can speak of <u>one</u> man because he is indivisible <u>qua</u> man even though in another respect he is divisible, viz. <u>qua</u> three-dimensional body.[10] Likewise in the case of mathematical quantities the parts separated from a divided line, for example, are actually ones, actually undivided, and are indivisible inasmuch as they can not be divided into parts which are each exactly the same length or larger than the original line. But each part is potentially divisible into smaller parts <u>ad infinitum</u> for "every continuous thing is divisible."[11] Also, in the case of number, the two 3's, for example, which can be

separated from 6 as its parts are "ones;" each is the one individual 3 that it is. Each 3 is indivisible only inasmuch as it can't be divided into other 3's (or higher numbers); in other words, each is indivisible qua 3 (similar to man's indivisibility qua man). But each 3 is certainly divisible into lesser numbers and ultimately into units.[12]

One important question which arises in regards to the ones of the parts of a quantity is this — can the parts of a quantitative whole be ones, indivisibles, of any kind whatsoever according to Aristotle? In other words, can they be substantial units, or qualitative units, or in general any nonquantitative units? Do ten individual colors, for example, each of which is a unit of quality, make up a quantitative plurality? We do refer to them by using a quantitative term, ten. And what about the prime movers which are also numbered, either as forty-seven or fifty-five? Taken as a group, do they form a quantitative plurality even when no one of them possesses any quantity? To resolve this question, we must take note of the Stagirite's distinction between the one which is quantitative and the one which is simply convertible with being and signifies its indivision. The Greek philosopher clearly differentiates between these two unities,[13] the one said of entities only in the category of quantity, the other said of entities in any category, and even of the prime movers which transcend all categories. He further distinguishes between the pluralities (and the numbers which count them) which are composed of these

different kinds of units.[14] A plurality of quantitative units (and their corresponding numbers) is not the same as a plurality of qualities, of actions, of prime movers, and so forth (and their corresponding numbers) whose units are not quantitative. In fact, pluralities (and numbers) which are not made up of units of the category of quantity — say, for example, a plurality of ten colors, or five actions, or forty-seven gods — would not be quantities.[15] It would follow, then, since mathematics for Aristotle is a science of quantity, that nonquantitative pluralities and their numbers are not within its scope.

However, there is no place where the Stagirite explicitly draws this conclusion, no place where he clearly applies his distinction between quantitative and nonquantitative ones to mathematics and the quantities it treats. And in fact, in speaking of the units and the pluralities composed of them which are studied in mathematics, he often neglects to specify that these units must be quantities.[16] Of course, we could simply pass off such passages by saying that he is not speaking precisely. However, I think there is more to it than that. I think it is more likely that Aristotle simply did not thoroughly think through his distinction between quantitative and nonquantitative ones with reference to the object of mathematics. (To look ahead, we will see in Part II that Aquinas' teaching is very explicit on this point.) Nevertheless, since he does clearly assert that mathematics studies quantities, and since pluralities composed

of nonquantitative units are not quantities, we can conclude, even though he doesn't, that such pluralities could not be objects of his mathematics; or rather that they could be treated by mathematics, only if they could be in some way translated into quantities.

I add this last remark because Aristotle does indicate that entities other than quantity can be treated as quantified. For example, he says that measure, which pertains properly to the category of quantity and involves determining the number of times a certain quantitative unit is taken, can also be extended to entities of other categories.[17] This implies that these other entities are in some way quantitative. He also speaks of qualities and other nonquantitative accidents as being quantities incidentally because, though in themselves they are not quantities, they are present in or related to that which in itself is a quantity.[18] Color, for example, though per se a quality, is incidentally a quantity inasmuch as it is the color of a surface which is a per se quantity. In what way these "nonquantitative quantities" can be objects of mathematics will be discussed later, in Chapter IV, when we speak of the branches of mathematics which study physical phenomena according to Aristotle.

Thus, concerning the oneness of each of the parts of a quantitative whole, the Greek philosopher's doctrine, though not perfectly explicit, seems to be that the parts must themselves be quantitative units, indivisible in some way, or reducible to quantitative units, if the whole which

they make up is to be itself a quantity or at least incidentally quantitative.

By also referring to the parts of the quantitative as by nature "this," Aristotle expresses much the same notion that he does in calling them "one." Indeed, in one text he practically equates "one" with "this."

> "To be one" means to be indivisible, being essentially a "this," and capable of being isolated either in place, or in form, or thought.[19]

The term "this" is used by the Stagirite to refer to that which is in some way, either singularly or specifically, one; it may, but need not, refer to an actual singular substance. It applies to that which is definite, determinate, actual rather than potential. Form, for example, is said to be a "this" and to cause the composite substance to be a "this."[20] As the above text states, that which is a "this" is an indivisible able to be isolated "either in place, or in form or thought." Note especially that a "this" may be a one capable of being isolated only in thought; the substantial form of a sensible thing, for example, cannot be physically separated or isolated from the composite thing. However, since this form by its very nature has an intelligible content of its own apart from the matter and the composite, and so has a unity, a oneness, in thought, it is called a "this" by Aristotle.[21] In like manner, the parts of a quantity may be called "this-es" even if they are only able to be divided from (isolated from) the quantity in thought, so long as by their nature they are determinate

unities mentally separated from each other. A number, for example, is a quantity divisible in thought into parts which by nature are ones, are "this-es," isolated in thought from each other. Six is mentally separable into parts three and three each of which is by its very nature as a number a determinate one isolated in thought from the other.

In general, then, this analysis of Aristotle's notion of "one" and "this" has shown that for him the quantitative in the strict sense (leaving aside incidental quantity) is that which is divisible, at least in thought, into parts each of which is by nature a definite, determinate, actual quantitative unit indivisible in some way.

The foregoing discussion will help us answer the first question we asked about Aristotle's description of the quantitative. Recall that we noted that he describes it in terms of potency, "that which is divisible," and we asked what corresponding actual features, called actual quantity, must be possessed by the quantitative in order for it to have this potentiality, this divisibility into parts? From the previous analysis, it seems clear that the divisibility which is the mark of the quantitative presupposes the possession of parts which are potentially determinate quantitative units, parts which are able to be separated at least in thought from the quantitative whole and from each other. As present in an actual attribute of quantity these parts are only potential units, for the quantity which they make up is itself an actual unity. (See pages 28-29.) Quantity, then, for Aristotle is an actual unity of quantitative parts. It is that which accounts for something

having parts "outside of" parts.(where the quantitative parts need not be parts extended spatially, like those of a block of wood, but may be parts only separable, only "outside of" each other, in thought). Things like blocks of wood which are composed of extended parts are divisible into determinate quantitative units because of their actual extension, their actual "spreadoutness" in space, their having parts spatially distinct. It follows that it is their characteristic of being extended that is their actual feature of quantity. On the other hand, a plurality, such as a number, whose parts are distinguishable but not spatially so, is quantitative, i.e. divisible into determinate quantitative units, because of its very plurality, because it is a unity of many quantitative parts. Hence it is this actual feature, the actual plurality, that is its actual quantity.

In general, then, the actual attributes of quantity of that which is quantitative (be it one extended thing or one plurality) is its possessing actually unified quantitative parts in some way (at least in thought) distinct from each other. To say it another way, the quantity, that is the actual feature which accounts for the divisibility of that which is quantitative, is roughly its magnitude in the case of spatial extension, or its plurality, in the case of number.[22] (These divisions correspond to the two basic species of mathematical science: geometry, the science of magnitude, and arithmetic, the science of quantitative plurality.)[23]

A more complete insight into Aristotle's notion of

quantity is attained when we contrast it with the closely related category of quality. Quality, he says, in the Categories, is that in virtue of which things are said to be such and such.[24] This description is very general; certainly quantity too is that in virtue of which things are said to be "such and such," e.g. such a size, such a plurality, such a magnitude. In order to be more precise, he goes on to say that quality is divided into subclasses:[25] one sort is habits and dispositions; another capacities and incapacities; another affective qualities (including sensible qualities) and affections; another form or figure; and in the Metaphysics he includes the differentia of the essence.[26] He adds in the Categories, "there may be other sorts of qualities"[27] than those classes just listed. It is clear, then, that quantity, as distinct from quality, does not refer to the above listed features of things. It is not quantity which accounts for things having certain habits or dispositions, capacities or incapacities, affective qualities, affections, figures, essential differences; it is quantity, which accounts for things and their attributes having a certain magnitude or being a quantitative plurality, and it is only this which is due to quantity.

An example may help. The attribute of quantity possessed by a man does not refer to his color, texture, or other sensible features, nor to his habits or dispositions, nor to his capacities or incapacities, nor to his shape, nor to his essential difference which makes him a man; all these features are qualitative. His quantity is his magnitude or

extension, his being "spread out" in space, for it is this which accounts for his having parts into which he is divisible. Put it this way. Man qua man is not a quantity, nor is man qua colored (affective quality), or qua courageous (habit), or qua sentient (capacity); man qua magnitude, qua spatially extended is. It is not because a man is a certain kind of thing or of certain sensible qualities, nor because he has certain capacities or habits or dispositions or a certain shape that he is divisible into quantitative parts, but rather it is because he has a certain magnitude or extension. And indeed, as we said earlier, it is just because he has such magnitude that his color, for example, which is spread over this spatial magnitude is also divisible into quantitative parts and hence is called by Aristotle a quantity incidentally.

The above example refers to quantity in the sense of magnitude. Let us take an example dealing with quantity in the sense of plurality, for example, six men. The quantitative attribute six does not refer in the case of this plurality to their sensible affections, nor their capacities, nor habits, nor dispositions, nor to their shapes, nor to their specific nature as men but only to their actual plurality, the actual unity they have as many quantitative parts, their actual manyness which is divisible into quantitative parts. To be sure one could speak of a plurality, a number, of colors or of habits (e.g. moral virtues) or of capacities (e.g. sense powers), or of shapes. But if the number in these cases refers to them as a plurality of colors as such, or of habits as such, or of capacities as such, or of shape as such, then

it is not a quantity because as such none of these is a quantity. (This was the point made earlier concerning the difference between quantitative and nonquantitative pluralities and numbers — pages 4-6.) Only insofar as these qualities are present in quantitative units could their plurality and number be quantitative. Just as the color which covers a surface is quantitative (i.e. divisible into quantitative parts) not qua color but qua spatially extended by the quantitative surface in which it is, so also six colors, or habits, etc., are quantitative (are divisible into quantitative parts) not insofar as they are a plurality of colors, nor a plurality of habits, but only inasmuch as they are in a quantitative plurality, in six quantitative units of some sort.

Let us now draw to a conclusion this discussion of the nature of quantity according to Aristotle. We have seen that quantity can be said to be the actual magnitude, the extension, of a thing and the actual plurality in the case of a unity of quantitative units. Quantity is the actual unity of quantitative parts. Because of quantity, i.e. because of magnitude and plurality, that which is quantitative is divisible, at least in thought, into parts each of which is some determinate kind of quantitative unit. Strictly speaking, quantity is either a unity of extended parts, that is a magnitude, or a unity of nonextended quantitative parts, that is a plurality, nothing more. It is not the substantial difference of a thing, nor its sensible features, nor any other qualitative characteristics, though in some cases they

may be incidentally quantitative.

One last point. It would seem to follow from what Aristotle says about the nature of quantity that this attribute is closely related to matter. Recall that he has described quantity in terms of potency; to be quantified is to be divisible, he says. Now the Stagirite teaches that the principle of potency in things is their matter.[28] Of course, the quantity of a thing or of a group of things is an actual and not potential feature: in the case of a thing its actual extension, its actual "spreadoutness"; their actual plurality, in the case of a group of quantitative units. Nevertheless the fact that a thing or a group is quantified means not only that a thing actually has magnitude or that a group has plurality but also that each has the capacity to be divided into quantitative parts. In other words, actual quantity brings with itself, so to speak, potentiality in the form of divisibility. But since potentiality for Aristotle is rooted in matter, then quantity demands matter.

It is fairly easy to see that this is so in the case of continuous quantity or magnitude. Because the magnitude a thing possesses is an actual spatial extension of its parts, this magnitude makes the thing really divisible into these parts. For the thing to have this potentiality, it must have a principle of potency which is matter. The case of the plurality which is discrete quantity is more difficult. Does its divisibility into quantitative units demand matter? The answer would appear to be that it does, for Aristotle seems to hold that discrete quantity arises through the

division of continuous quantity.[29] Therefore, if continuous quantity demands matter so ultimately would discrete. (Of course, a plurality which is not a discrete quantity, for example a plurality of qualities or of prime movers, would not as such demand matter for their divisibility comes simply from the fact that they are actually diverse forms, accidental in one case and substantial in the other. Only a quantitative plurality, a discrete quantity, would seem to demand matter since, as we said, Aristotle apparently holds that it arises through the division of continuous quantity and continuous quantity demands matter.)

Actually, this general relation of quantity to matter is not at all developed by the Greek philosopher. In fact, I know of no place where he explicitly makes the connection between the two; though as I have tried to show it does seem to follow necessarily on the basis of his description of quantity.[30] (To look ahead, Thomas Aquinas will explicitly make the connection.) Aristotle does speak of the peculiar matter that is present in mathematical quantities, and this we will discuss at length in the next chapter. But even here in the case of the relation between mathematical quantity and its special matter, we will find that the Stagirite's teaching is often vague and incomplete.

FOOTNOTES

[1]Aristotle, Metaphysics translated by W.D. Ross in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), XI, 3, 1061b 20-23. Unless otherwise indicated, all future citations of Aristotle refer to the translations in The Basic Works of Aristotle. See also XIII, 2, 1077a 22-23 where Aristotle refers to

"the subjects of mathematics, which are divisible and are quantities . . . ."

[2]W.D. Ross, Aristotle (New York, 1962), p. 23. See his discussion of the categories in pages 23-25.

[3]Metaphysics, V, 7, 1017a 23-28.

[4]See his Posterior Analytics, translated by G.R.G. Mure in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), II, 13 and Metaphysics, VII, 12 where Aristotle talks about the process of definition.

[5]Metaphysics, V, 12, 1020a 8-9. I have substituted the phrase "the quantitative" for the term "quantum" used in Ross's translation. The former phrase is, I think, a slightly more accurate rendering of the Greek. On this point, see Joseph Owens, An Elementary Christian Metaphysics (Milwaukee, 1963), pp. 165-166 and his article "The Aristotelian Conception of the Sciences," International Philosophical Quarterly, IV (1964), 209.

[6]Metaphysics, V, 6, 1016b 4-5.

[7]Metaphysics, X, 2, 1053b 24; note especially the lines following where he speaks of the one in every category.

[8]Metaphysics, X, 1, 1052b 15-16.

[9]Aristotle discusses the different meanings of one, and hence of indivisible in Metaphysics, X, 1. In one place he states, "But it is not in the same way that every 'one' is indivisible. . . ." (1053a 22)

And he goes on to distinguish ones which are absolutely indivisible and ones which are indivisible not absolutely but in some respect.

[10]Metaphysics, XIII, 3, 1078a 23-29.

[11]Metaphysics, X, 1, 1053a 23. In the Physics, translated by R.P. Hardie and R. K. Gaye in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), VII, 1 and 2, the Stagirite explains how everything continuous is composed of infinitely divisible parts.

[12]Aristotle states "that which is one is indivisible, either absolutely or qua one," Metaphysics, X, 1, 1053b 8. I take this to mean that an entity which is one is indivisible insofar as it is one. Thus, one man is indivisible qua one man; he is not divisible into little men. Likewise one number 3 is indivisible qua 3; it can't be divided into little 3's. One line is indivisible qua the one line it is; it can't be divided into lines of the same length as itself.

[13]In the Metaphysics, X, 2, 1053b 24 ff. and XIV, 1, 1087b 34 ff. Aristotle clearly differentiates between the different ones in each of the different categories.

[14]In Metaphysics, X, 1, and XIV, 1, he discusses measure and states that the one, the basic unit of measure, and hence the number which is a plurality of ones, is different in each category.

[15]Greenwood, Études sur . . ., seems to make the mistake of thinking discrete quantities (numbers) are composed of units which are not quantitative but equivalent to the units of being, p. 38.

[16]Some places where Aristotle speaks of arithmetic studying the unit and numbers made therefrom and does not say that these units must be quantified are: Metaphysics, XIII, 3, 1078a 23-29 and Metaphysics, XIII, 8, 1084b 24-31. In fact, in one place he seems to clearly identify the unit of mathematics with substantial unity, Posterior Analytics, I, 26, 87a 33-37:

> "A science like arithmetic, which is constituted of fewer basic elements, is more exact than and prior to geometry which requires additional elements. What I mean by additional elements is this: a unit is substance without position, while a point is substance with position." [underlining mine]

Yet, in other places, he appears to distinguish between the unit of mathematics and the unit convertible with being — for example, Metaphysics, III, 4, 1001a 25-27.

[17]Metaphysics, X, 1, 1052b 18-31, X, 2, 1053b 4-8, and XIV, 1, 1087b 34 - a 14. On this point, also see the article by Mark Heath, "Can Qualities be Measured?",

Thomist, XVIII (Jan. '55), 34 ff., 50 ff.

[18]Metaphysics, V, 13, 1020a 14-33.

"[S]ome things are called quanta in virtue of their own nature, others incidentally; . . . Of things that are quanta incidentally, some are so called in the sense in which it was said that the musical and the white were quanta, viz. because that to which musicalness and whiteness belong is a quantum, and some are quanta in the way in which movement and time are so; for these also are called quanta of a sort and continuous because the things of which these are attributes are divisible."

[19]Metaphysics, X, 1, 1052b 15-18.

[20]For Aristotle's meaning of the term "this," see the whole of Father Owens' excellent analysis in Chapter 13 of his The Doctrine of Being in the Aristotelian Metaphysics (Toronto, 1957). He has reference there to all of the pertinent passages of Aristotle.

[21]Metaphysics, V, 8, 1017b 25-26. For a discussion of Aristotle's doctrine on the form as a "this," see Owens, The Doctrine of Being. . . ., pp. 238-240.

[22]Metaphysics, V, 13, 1020a 10-14.

[23]Rhetoric, translated by W. Rhys Roberts in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), I, 2, 1355b 30-31.

[24]Categories, translated by E.M. Edghill in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), 8, 8b 25.

[25]Categories, 8. For a detailed explanation of these classes, see Owens, . . . Christian Metaphysics, pp. 170-178.

[26]Metaphysics, V, 14, 1020a 33; b 14-15. Actually I have great difficulty in seeing how the differentiae of the essence could be a quality. If it causes a thing to be a particular kind of thing, it would have to be of the category substance. See my discussion of this in footnote 13 of the next chapter.

[27]Categories, 8, 10a 25-26.

[28]See Book I, Chapters 7 and 9 of the Physics where Aristotle sets forth the basic principles of physical things: matter, form, and privation — matter referring to that which is related to form as potency to act. See also Chapter 3 of Book II of the Physics where he discusses each of

the four causes. Ross, Aristotle, has a good discussion of this point in his third chapter in the section entitled "The Four Causes," especially, p. 76. In some places, Aristotle explicitly identifies matter and potency; for example, in the De Anima, translated by J. A. Smith in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), II, 1, 412a 10:

"Now matter is potentiality, form actuality . . . ."

[29]Physics, III, 7, 207b 10-14. See also On Generation and Corruption, translated by Harold H. Joachim in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), I, 2, 316b 28-29. On discrete quantity as subsequent to continuous, cf. Owens, . . . Christian Metaphysics, pp. 167-168.

[30]There is one very obscure text in which the Stagirite may make the connection between quantity and matter. In Metaphysics, XI, 6, 1063a 26-28, he states,

"But essence depends on quality, and this is of determinate nature, though quantity is of indeterminate."

Father Owens, in his The Doctrine of Being . . ., p. 396, n. 10, interprets this passage as saying that quantity "is based on indefinite nature," i.e. is based on matter. However, that this is what Aristotle is saying is not at all clear to me. In Metaphysics, VII, 1, 1029a 11-19, Aristotle does give a slight indication that quantity is the accident closest to matter.

On quantity as demanding matter, Owens, . . . Christian Metaphysics, p. 167 says:

"A corporeal nature just in itself, therefore, is not extended. But it is a nature that is capable of extension in the three dimensions. The capability or potency to extension belongs to its very essence, ... A material nature is necessarily composed of the two physical principles, matter and form. The matter allows the formal principle to be spread through parts outside parts without any formal increase whatsoever. In this way it renders the nature itself potentially extended. But that potentiality can be actuated only through an accident, the accident of quantity. Not even the matter, just in itself, is extended. The matter is only the substantial basis of extension, insofar as it makes a formally identical nature capable of spread by a superadded accident through parts outside parts."

CHAPTER II

THE PRECISE NATURE OF THE QUANTITIES WHICH ARE THE OBJECT OF MATHEMATICS

The Difference between Mathematical Quantities and Quantified Physical Things

Now that we have discussed at some length the nature of quantity, the object of mathematics, we should attempt to become more precise as to the type of quantity which according to Aristotle is studied by mathematics. One thing is clear. Mathematics according to the Stagirite does not treat quantified <u>things</u> of this world, i.e. quantified physical things.

> . . . with what sort of things must the mathematician be supposed to deal? Certainly not with the things in this world; for none of these is the sort of thing which the mathematical sciences demand.[1]

The very nature of the quantities studied in mathematics shows that they are not such things. For one thing, Aristotle points out that mathematicals are not sensible, that they lack all sensible qualities.[2] Clearly, a mathematical figure and other magnitudes treated in the science of geometry have no color, no odor, no texture, no temperature or flavor, no hardness or softness; they emit no sound. This is even more true of mathematical numbers. These quantities, which are treated in arithmetic, obviously have no color, odor, flavor,

etc. — in short, no sensible qualities. A simple glance at the arithmetic and the geometry of the Greeks (and even of our own day) will thus bear out Aristotle's point.[3] The numbers and magnitudes which each respectively deals with have no sensible qualities.

A further difference between mathematical quantities and quantified physical things is that the former in many cases are not three-dimensional while a physical thing "has being in every dimension."[4] A mathematical number, for instance, has no dimensions (It is divisible ultimately into dimensionless quantitative units.);[5] a plane surface, and hence plane figures, have only two dimensions (They are divisible in length and width and have no depth.).[6]

Probably one of the most significant differences in Aristotle's mind between the quantities treated by mathematics and quantified physical things is the immobility, and, therefore to an extent, the immateriality, of the former. The Stagirite states, "[M]athematics is theoretical and is a science that deals with things that are at rest;"[7] and again he says, "[T]he objects of mathematics . . . are of the class of things without motion."[8] Physical things, on the other hand, are defined precisely as having within themselves a source of movement.[9]

Now by the term motion, used in its strictest sense, Aristotle means three and only three types of change: change in quality, change in quantity, and change in place. Substantial change, generation or corruption, while it certainly

is change, is not motion in the strictest sense of the word for it does not consist of a process during which a potential is actualized but is rather instantaneous.[10]

Certainly mathematical quantities cannot undergo the type of motion known as alteration, i.e. change of quality. For alteration, according to Aristotle's analysis in the Physics,[11] consists, in the precise sense, in change of sensible quality only; the changes in the other species of quality — habits, capacities, etc. are not alterations or motion strictly speaking. Clearly, then, mathematical quantities, since they lack all sensible qualities, cannot undergo motion in the category of quality. It might seem that some mathematicals, viz. geometrical figures, could undergo alteration by change of their shape or figure, for shape or figure is listed as a species of quality. However, such a change would not be motion but would be analogous to substantial change, which, as we said above, is not motion strictly speaking. A triangle, for example, which changed its shape to rectangularity would simply cease to be a triangle and become a rectangle; no motion is involved, only change of the mathematical essence. In general, since Aristotle holds that the shape of a geometrical figure is what accounts for its being the kind of figure that it is[12] (and for this reason I do not think that such shape is really a species of the category quality for the Stagirite)[13] any change of shape of a mathematical figure would be analogous to a substantial change, and hence would not be motion.

Similarly any change in what Aristotle calls the qualitative features of a number would also be analogous to substantial change of that number and hence would not be motion. Such features as the capacity of a number to be divisible by another, or one to be the square of another, are called "qualities" of numbers by the Stagirite.[14] Now the only way a number could change in such "qualities" is by ceasing to be the number that it is. For example, 9 can't cease to be the square of 3, nor 10 divisible by 2, nor 27 the cube of 3 unless 9 ceases to be 9, 10 to be 10, 27 to be 27, unless they respectively become other numbers. Therefore, a mere change of number in quality alone, i.e. by alteration without "substantial" change, is impossible. Thus, to repeat our main point, mathematical quantities are unable to undergo alteration, to be moved in respect to quality.

Since mathematicals are quantities it might seem that increase or decrease, change in quantity, would be a type of motion which they could undergo. A moment's reflection, however, will show that this is not the case. A given mathematical cannot be increased or decreased in size and still remain the particular mathematical that it is. For instance, if one side of a particular geometrical figure, say a triangle, is enlarged, the result is not that the original triangle is larger (an accidental change in quantity); the result is a completely different triangle. It is true that the original triangle has become another triangle, not a specifically different figure. In other words, there has

been no change in kind. Nevertheless, the original triangle has still ceased to be for a particular figure is composed only of lines or surfaces of definite dimensions. Increase or decrease these lines or surfaces and the original figure simply is no more but a new one has come to be. Likewise in regard to numbers, if a part, no matter how small, is added to or subtracted from a certain number, the result is not that the original number is larger or smaller; the result is a completely different number.[15] All that a number is is a certain plurality of units.[16] Change that plurality by adding or subtracting even the smallest unit and the original number ceases and a new number, a new plurality of units, comes to be. Clearly then, mathematical quantities cannot undergo the accidental change of increase or decrease, motion in respect to quantity.

The third kind of motion, locomotion, is also impossible for mathematical quantities since according to Aristotle they are in no place: "[M]athematical objects are nowhere . . . ."[17] One can ask — where is Mount Olympus? — or some other physical thing. But to ask where is a triangle or where is three is to ask a meaningless question. And yet, in relation to one who is thinking of them, mathematical figures at least do clearly have various positions. (Numbers have no place for they have no extension.) For example, the geometer may consider a triangle and a circle placed relatively to each other thus: . In fact, the geometer, both in Aristotle's time and in our own day, considers his objects

in all kinds of various spatial relations — some intersecting, some tangent to others, some inscribed in others, some circumscribed, etc. However, according to Aristotle, these spatial arrangements are places that geometrical objects have only "in thought";[18] they do not have by nature any real place or position. He makes this clear in the Physics.

> [T]hough they do not have their being in a place, they nevertheless have distinctions of position (like "right" and "left") relative to us; they have this "position" therefore in concept [i.e. in thought] only, but they do not have any position in their own nature.[19]

This passage is from Book IV in which Aristotle discusses place. His whole discussion there indicates that he conceives place in reference to real physical things having a relation of container and contained. Place for him seems to refer to the real container of that which is in place, or more precisely to "the boundary of the containing body at which it is in contact with the contained body."[20] Change of place, locomotion, therefore, involves the change from one real container to another. But if mathematical bodies have no real place, as he has said (and furthermore they are not real bodies nor actually in real bodies as we shall see later), then they cannot undergo locomotion.

What about the "apparent" locomotions that mathematical objects seem to undergo in the thought of the geometer? To cite one clear example of such locomotion — to prove the congruency of two figures involves taking

one and "moving" it so that it is placed on top of another, if they are congruent they will coincide perfectly.[21] Why can't such changes of place in the thought of the mathematician be called locomotions?

The reason is that, in spite of the above example, mathematical figures actually cannot change accidentally in place. This can be seen by starting with the most basic element of such figures, the point. A point is defined by Aristotle as a unit having position; one point is distinguished from another precisely by its position.[22] To put it another way, a point is individuated by its position. Now to move one point from its position to another position is simply impossible. At the other position there is another point, for it is by position, as we said, that points are individuated.[23] Therefore, a point cannot change in place and remain the same point as it was originally. But the Stagirite teaches that points are the extremities, the limits, of lines.[24] If points cannot change accidentally in position, then neither can lines. A line is defined as a limited length; the limits are its points.[25] If a line were to change in place it would thereby change its limits for it would change its terminating points and thus it would not change accidentally but would become a different line. Therefore, a line too cannot change accidentally in place. But if a line cannot, then neither can planes for lines are their limits; and if planes cannot, neither can bodies for planes are their limits.[26] Thus

none of the elements of mathematical lines, figures, and solids, nor therefore the lines, figures and solids themselves, can change accidentally in place.

Thus, we have seen in detail why Aristotle calls mathematical quantities immobile.[27] We have seen that they cannot undergo the three types of motion he lists — change in quality, change in quantity, change in place. As mentioned above, substantial change, generation and corruption, is not motion in the strictest sense of the term.

I emphasize this last point because some commentators say that the Stagirite holds that mathematicals are unchangeable, which would mean they are free from generation and corruption as well as the motions we have mentioned.[28] However, I know of no place where Aristotle says they are unchangeable; what he says again and again is that they are immobile.[29] Indeed, he himself refers to the generation and corruption of mathematical quantities; and he actually uses the phrase "generation of the objects of mathematics. . . ."[30] By generation and corruption in reference to mathematicals I do not mean, of course, that proper to physical things. What I mean is that some mathematicals come to be, i.e. are generated, from others, which in turn are corrupted. The following will, I hope, make this clear.

The Generation and Corruption of Mathematical Quantities

Generation, according to the Greek philosopher, is distinguished from simple coming to be for generation is

coming to be from something. Likewise, simply ceasing to be is not corruption for corruption involves becoming something.[31] Now in regards to numbers, he states, "[F]rom many numbers one number is produced,"[32] apparently referring to operations of addition and multiplication. Let us take a simple example of the generation and the corresponding corruption of numbers. Two 3's can be combined by addition to make up one number 6. After the addition the 3's are no longer actually two individual numbers, only the one 6 which came from them is actual. This 6 is not simply an aggregate of the two numbers which make it up (if it were, no new being would have come to be and so no generation would have taken place). It is one individual number unique in its own right. Aristotle states, "[T]he number is a unity and not like a heap."[33] And he stresses this unity in saying that, "6 is not what it is twice or thrice [i.e. is not twice 3 or thrice 2,] but what it is once; for 6 is once 6,"[34] emphasizing that 6 is not a mere aggregate of lesser numbers. Thus in the example given, one entity, 6, has actually come to be, been generated, from the two previously actual entities, 3 and 3, and these two numbers in turn have been corrupted. Of course the 6 in its turn could be corrupted by division into the two 3's which are potentially in it. This corruption would entail the actual generation of these two unities, these two individual numbers. The Stagirite refers to numbers as divisible into smaller numbers and ultimately into units which are in them potentially and are actualized, are generated in other words,

by the division.[35]

In the case of continuous quantities Aristotle refers to the semicircle and segments of a circle as being potentially in the circle and being actualized (generated) by its division; he says the same about the halves of a line and the acute angles which are divisions of the right angle. He also speaks of the corresponding generation of these figures from other figures which become their parts.[36] An example is useful to illustrate the generation and corruption in this case too. Two actual lines, each of which is one continuous quantity, a unity, in its own right, can be joined end to end to form one line. The two lines cease to be actual continuous quantities (are corrupted in other words) and become the parts of the one longer line which comes to be (is generated) from them. This longer line is not simply an aggregate of the two shorter ones but a new unity, a new individual continuous quantity. Of course, it can in turn be corrupted by division so that the two original lines are generated.

Thus it seems clear that the Stagirite does hold that at least some mathematical objects are generated from others, and correspondingly some corrupted into others.[37] I say "some" mathematical objects because it is certainly true that the unit and the point, since they are indivisible and hence simple, cannot be generated or destroyed. Generation and destruction can only take place in that which has parts.[38]

In fact, simply because all the objects of mathe-

matics are quantities, i.e. have parts distinct from parts, and are divisible in some way or other (as we saw in Chapter I), they must be able to be generated and corrupted. Every individual mathematical unity, be it a number or a figure, can generate other mathematical quantities simply by being corrupted into the parts which make it up. By division these parts become actual mathematical unities, where before as parts of the whole they were only potential.[39] In like manner every individual mathematical unity can be generated by the combination of certain other quantities which lose their own individual unity in becoming parts of the larger quantity.

Along the same line, the fact that Aristotle refers to the parts of mathematical figures (e.g. the segments of a line or circle, the acute angles into which a right angle is divisible) as their matter, and the parts (units) of numbers as their matter,[40] indicates that he considers them to be that out of which the composite mathematical is formed and that into which it can be broken down.[41] In our earlier example we spoke of joining two lines end to end to get one line their combined length. As its elements the two original lines would be the matter of the new one and they would be its matter inasmuch as the new line could be divided into them making them actual again. Likewise, two semicircles of the same size which can be joined to form one circle and into which this circle is in turn divisible would be its matter. In the case of number, the units which make up a

number and into which it is divisible would be its matter. Incidentally, this matter which is present in mathematical quantities and which he identifies with their parts is clearly distinguished by Aristotle from the matter of sensible things. He calls the matter of mathematicals "intelligible" rather than sensible matter, apparently because mathematicals are not sensible as we saw above.[42] We will see much more of this "intelligible" matter as we proceed in our analysis of mathematical quantities.[43]

To sum up this point, then, it seems clear that for Aristotle mathematical quantities are immobile in the strict sense of the word and not unchangeable. I know of no place where he denies that they are generated and corrupted; instead, as we have seen, he refers often to the generation and corruption of lines, figures, angles and numbers. Furthermore, the very fact that quantities are defined in terms of divisibility and have parts distinct from parts means that they can be composed from other quantities and resolved into them. Finally, the fact that he calls the parts of mathematical quantities their <u>matter</u> indicates that such quantities can be formed from them and broken down into them, i.e. generated and corrupted.[44] (I should emphasize here that the terms generation and corruption used in reference to mathematical objects are actually used in an analogous sense. To look ahead, we will treat in detail the fact that mathematicals are not actually real beings. Since this is so, then of course they are not really generated and corrupted

as physical beings are.[45] However, as the foregoing analysis has shown, in their own realm — and exactly where that is we have not yet specified — mathematicals do come to be from and are corrupted into each other similar to the way physical things are generated and corrupted.)

Now I realize that in calling mathematicals immobile, Aristotle could be using the term, as he sometimes does, in a general and less precise sense which excludes generation and corruption as well as the other motions.[46] And there is a sense, but only a sense, in which mathematicals are not generated or corrupted. We noted above that Aristotle holds that generation and corruption can only take place in that which has parts. This means that no form is per se generated or corrupted. He states,

> [N]o one makes or begets the form, but it is the individual that is made, i.e. the complex of form and matter that is generated.[47]

He points out that if the form itself were generated and corrupted it would not be just form but would have to be composed of a material principle. The whole of Chapter 8 in Book VIII of the Metaphysics discusses this; at the conclusion of one of these discussions he says,

> It is obvious, then, from what has been said, that that which is spoken of as form or entity is not produced, but the concrete thing which gets its name from this is produced, and that in everything which is generated matter is present and one part of the thing is matter and the other form.[48]

Both of these passages show clearly that it is only per accidens that a form is generated and corrupted, i.e. only inasmuch as it is part of the form-matter composite which

is generated and destroyed.

This point can apply to mathematical as well as other entities. The forms of mathematicals are not per se generated and corrupted but only per accidens inasmuch as they are part of individual mathematical composites which are generated and corrupted. For instance, the form of circle, call it circularity, which accounts for two semicircles as matter making up one circle, is not per se generated and corrupted but only per accidens, inasmuch as this or that individual circle in which it is is generated and corrupted. Likewise, the form of nine, nineness, which accounts for nine units not being simply an aggregate but making one number nine, is not per se generated or corrupted; it is per accidens inasmuch as it is in this or that individual nine. In this very restricted sense, then, mathematicals, meaning the form of these quantities, are not generated or corrupted and hence are immobile in the broadest sense of the word.

Now it is true that a science which studies form-matter composites primarily studies the form, according to Aristotle. This is so because, as he points out, a composite is primarily its form inasmuch as the form is the intrinsic actuality which causes the composite to be the one particular entity it is.[49] It follows, then, that mathematics would study primarily the form of mathematical quantities and in this sense be primarily, though not exclusively, of that which is in a restricted sense, viz. per se, ungenerable and incorruptible.[50] In this way, the primary

object of mathematics could be said to be immobile in the broad sense.

Actually, some commentators go so far as to assert that mathematics for Aristotle deals with form and only with form, not with form-matter composites at all.[51] If this is so, then there would be even more reason for saying that the objects of mathematics, are ungenerable and incorruptible. As a matter of fact, Aristotle does state in his Posterior Analytics that "mathematical sciences concern forms."[52] Does this mean that the objects of mathematics are really only forms? (Needless to say, any time I speak of forms as objects of mathematics for the Stagirite I mean quantitative forms, not the substantial forms of natural things, for we have seen his view that mathematics studies quantities.) Or could his statement be interpreted to mean no more than what we said above; viz. mathematics deals primarily but not exclusively with forms inasmuch as a particular mathematical, like every form-matter composite, is primarily its form? This question is crucial for the way we answer it will determine what is the true object of mathematics for Aristotle.

The Central Problem — the Essences of Mathematical Quantities; Pure Forms or Composites of Form and Matter?

Before we attempt to answer the above question, it would be well to pause and review the progress we have made in determining the exact object of mathematics. From the beginning of this chapter, we have been attempting to de-

fine more precisely the particular type of quantities with which mathematics deals. We have pointed out that according to Aristotle mathematics does not deal with the things of this world. In expanding on this point we have shown that, according to the Greek philosopher, mathematical quantities differ from quantified natural things inasmuch as the former are not sensible, are not in many cases three-dimensional, and finally are not mobile in the strict sense, i.e. cannot undergo alteration, increase and decrease, and locomotion. We showed, on the other hand, that for the Stagirite individual mathematicals can be generated and corrupted. The question was then raised as to whether or not there is any sense in which he might say that the objects of mathematics are ungenerable and incorruptible (and hence immobile in the less strict meaning of the term). We answered that there is a restricted sense, if by the object of mathematics is meant not the individual mathematicals, which are composites of form and matter (where the matter is intelligible matter), but rather their form. For form, Aristotle says, is not per se generable or corruptible, but only per accidens insofar as it is part of a composite. In connection with this, we noted that a science which treats of form-matter composites does according to the Stagirite have primarily, though not exclusively, the form for its object. Therefore, we concluded that mathematics too would be primarily of forms and so, primarily, though not exclusively, of what per se is not generated or corrupted.

In pursuing this notion of form as the primary object of mathematics, we pointed out that Aristotle himself actually says in one place that mathematical sciences concern forms, and we noted further that some commentators hold that this is his doctrine, viz. that it is form alone, not composites of form and intelligible matter, which is the object of mathematics. This is the problem that we now want to resolve for it is crucial in determining the precise object of mathematics. To put it in the form of a question — does mathematical science for Aristotle consider only forms, pure quantitative forms; or does it rather deal with form-intelligible matter composites, and hence primarily, though not exclusively, with form? Put it this way. Up to now we have been simply identifying the quantities studied by mathematics with mathematicals which are actually composites of form and intelligible matter, such as triangles, circles, numbers, etc. Certainly, Aristotle himself does cite such entities as objects of mathematics. Perhaps, however, this easy identification is imprecise. Perhaps what the Stagirite holds is that strictly speaking mathematics is not of these composites as such but only of the quantitative forms which they contain.[53] This is what we must determine.

We could ask a similar question about the essence of mathematical quantities. Aristotle's position that science does not deal with the individual as such is well known. "[E]very science is of universals," he says.[54] Therefore, mathematical science does not consider individual mathemat-

ical quantities, individual composites of form and intelligible matter as such, e.g. this triangle, this circle, this number 9. Rather mathematics treats of the essence or nature present in the individual mathematical, the essence triangle for example, just as physics considers the essence man, not the individuals John and Paul as such.[55] And now the question — do the essences of mathematical quantities, the essences which are studied by the mathematical science, consist according to Aristotle of forms alone, pure quantitative forms, or of form and intelligible matter (where the matter would not, of course, be individual intelligible matter, but what will by later men be called common intelligible matter)?

Before undertaking the detailed analysis necessary to answer this question, it would be well to anticipate what might be offered as an obvious answer. It might be argued that the very fact that mathematicals are quantities demands the presence of matter in their essence. Recall that in Chapter I (pages 14 and 15) we noted that Aristotle teaches that quantities of their very nature involve divisibility into parts. We then went on to say that only that which has a potential principle, viz. matter, can be divisible. Because of this, it might be proposed that the essences of mathematical magnitudes could not consist only of forms but would have to also include matter ("intelligible" matter, of course). And indeed, we have seen above that Aristotle does refer to the parts into which mathematicals

are divisible as their matter, viz. the segments of figures, the units of numbers. But is matter, meaning these parts, present in the essence of mathematicals? Certainly, it is only the individual, this circle or that line, that can be divided into parts, which is to say it is only the individual which is able to be corrupted and to generate other mathematicals. The essence "circle" or "line" is not per se divisible, no more than it is per se generable or corruptible. Does this mean that it is only the individual which must contain such matter? To be sure, I have stressed the fact that individual mathematical quantities have intelligible matter in order to show that they are generable and destructible (above, pages 30 and 31); but our question here is whether or not the essence of mathematical quantities includes matter, common (not individual) intelligible matter. Indeed, the following passages indicate that for Aristotle matter, meaning the parts of mathematical objects, is present only in individual mathematicals.

Almost all of the texts which follow are located in Chapter 10, Book VII of the Metaphysics in which Aristotle is seeking to determine whether or not the parts which are the matter of a thing are contained in the formula of that thing. In reference to the formula of circle, which in the following passage he contrasts with the formula of a syllable, he states,

> [T]he formula of the circle does not include that of the segments but the formula of the syllable includes that of the letters; for the letters are parts of the formula of the form, and not matter,

> but the segments are parts in the sense of matter on which the form supervenes.[56]

Clearly the only parts contained in the formula of the circle, as in the formula of a syllable, are those which are <u>not</u> material. He explains that that is why the formula of a syllable does include that of the letters while that of a circle does not include the segments. Notice, too, that the formula of a thing seems to be identified with the formula of the form. The formula of a thing, then, apparently includes only the form, not matter. Now the term "formula" in this particular context seems to be identified with definition.[57] But since Aristotle asserts elsewhere that the definition is a statement of the essence,[58] it would follow that the essence of circle consists only of form.

A little later on in the same chapter Aristotle speaks again of the circle and its matter.

> [M]aterials are principles and parts of the concrete thing, while of the form they are neither parts nor principles. And, therefore, the clay statue is resolved into clay and the ball into bronze and Callias into flesh and bone, and the circle into its segments; for there is a sense of "circle" in which it involves matter. For "circle" is used ambiguously, meaning both the circle, unqualified, and the individual circle, because there is no name peculiar to the individuals.[59]

This text speaks of a "sense of 'circle'" which "involves matter." Since its opening line states that "materials are principles and parts of the concrete thing," it is clear that the "sense of circle" which "involves matter" is the concrete individual circle. Now the individual

circle is clearly distinguished from "circle, unqualified." What does "circle unqualified" refer to? From its very first line, this passage is contrasting the concrete material thing with the form. Thus, "circle unqualified," which is clearly said to be distinct from the individual circle would have to be circle in the sense of form. If this distinction between "circle unqualified" and individual circle can be applied to the essence circle, we would have to say that certainly the essence would not be the concrete individual circle, the only one which Aristotle says involves matter. But his text gives only two meanings for "circle," "circle unqualified" — form, and individual circle — form and matter. Therefore, the only conclusion that can be drawn from it is that the essence circle would be "circle unqualified" which is identified with pure form.

A passage located in the next chapter in which the Stagirite again talks about the circle refers back to the one just given and supports this conclusion for it even more clearly indicates the identity of the essence circle with the form alone, excluding matter. The point is made by Aristotle that "there is some matter in everything which is not an essence and a bare form [Note the identity!] but a 'this'."[60] Now we noted earlier that the term "this" often means an individual (Chapter I, page 8); certainly what immediately follows the above passage indicates that that is its meaning here. He continues, "The semicircles, then, [which he says are the matter of a circle][61] will

not be parts of the universal circle, but will be parts of the individual circles, as has been said before . . . ."[62] The reference here is to the immediately preceeding chapter, where, as we saw, the matter, in the sense of segments of a circle, was said to be included only in the individual circles. Note the parallelism between this passage and the one cited in the previous paragraph. In this text as in the previous one matter is said to be part of the individual circle only. In this text matter is denied of the universal circle; in the former, it was denied of the circle unqualified. And the "universal circle" distinguished from individual circles seems clearly identified with the essence circle, which itself is called the "bare form." It follows, then, once again, the essence consists of form alone; no matter is included.

Note that in none of these passages is there to be found any notion of a "common" matter present in the "formula of the circle," in "circle unqualified," or in "universal circle"; matter without any qualification is explicitly denied of each of these and is included only in the concrete individual.[63] To sum up all three texts: the last one seems to identify essence and pure form with universal circle which lacks matter; the second one to identify essence with circle unqualified which is form and lacks matter; and the first to identify the formula (definition) of the circle with the formula of the form only.[64]

Let us leave the example of the circle and turn to texts in which Aristotle refers to a line, and apparently

denies that its matter (its half segments) is part of its essence, which is then identified with its form. Turning first again to Chapter 10 of Book VII of the Metaphysics, we find,

> For even if the line when divided passes away into its halves, or the man into bones and muscles and flesh, it does not follow that they are composed of these as parts of their essence, but rather as matter; and these are parts of the concrete thing but not also of the form . . . .[65]

The text seems quite clear. Matter is not part of the essence of line but only of the concrete line. The essence line is the form lacking matter. There is also an interesting text in the De Anima which, though obscure, seems to say much the same thing, viz. that the constitutive essence of the straight line, "straightness," does not include matter, while "what is straight," viz. the individual straight line, does.[66] The passage begins by noting the similarity between "what is straight," and "what is snub nosed."

> In the case of abstract objects [as we shall see this is clearly a reference to mathematical objects (page 63)] what is straight is analogous to what is snub nosed . . . .

The expression "what is snub nosed" was used in a preceding sentence and clearly meant the individual snubbed nose.[67] The passage continues:

> [F]or it [what is straight] necessarily implies a continuum as its matter; its constitutive essence is different, if we may distinguish between straightness and what is straight . . . .[68]

The only other mathematical object which is referred to in Chapter 10, Book VII of the Metaphysics is

the right angle. The formula of the right angle is said not to include the acute angles since they are matter and matter is included in the concrete individual. Once again matter seems to be excluded from the statement which expresses the essence of a thing, the same point we saw in regards to the circle.[69]

Does the essence of mathematical quantities consist of the form alone or of form and intelligible matter? The analysis of the foregoing texts indicates that Aristotle holds that in the case of mathematical magnitudes at least (and perhaps the same points would apply to numbers) their essence is simply the form and intelligible matter is included only in the concrete individual composite. Certainly, there is no mention of any notion like common intelligible matter which would be included in the essence of circle, of line, and of right angle. The only matter which is mentioned is relegated to the concrete individual.[70]

And yet, can we be sure that this is Aristotle's position? One thing is certain, these texts are very few and all of them are not perfectly clear. Furthermore, in opposition to the above texts there is at least one place where the Stagirite seems to include matter in the essence of mathematical quantity. In a chapter in his Posterior Analytics, where he is discussing and illustrating demonstrations through each of the four kinds of cause, Aristotle uses a mathematical proof as an example of a demonstration through material cause.[71] Now since mathematical

demonstrations (as we shall discuss in Chapter IV of this Part) are based upon the essence of the mathematical quantity, it would follow that mathematical essences must contain matter, in other words that matter is not limited simply to individual mathematicals. And yet, it is very interesting to note (as St. Thomas does in his Commentary),[72] that Aristotle never explicitly predicates the phrase "demonstration through material cause" of this mathematical demonstration, even though it is abundantly clear from the context that he is using it to illustrate just such a proof. Why doesn't he call it a demonstration through material cause? Could it be that he has not really decided whether matter is part of the mathematical essence? This seems quite possible, especially in the light of the earlier texts we have seen, but it is only a possibility.

Perhaps if we could find more material pertaining to Aristotle's position on this point, we could clear up any questions. However, more material can be obtained only if we make our problem more general. Aristotle devotes very little space to the question of the constitution of mathematical essences; however, he does spend a great deal of time, especially in the Metaphysics, seeking to determine the ingredients of the essences of physical things. If we determine whether the physical essence for Aristotle consists of form alone or of form and common matter, this could help us solve our problem concerning the mathematical essence. Of course, even if the ingredients of the essence of physical

things are determined, it will still rema[illegible]
what is said about it can also be ap[illegible]
mathematical quantities since the la[illegible]
from physical things. However, in t[illegible]
shed some light on the makeup of the [illegible]
mathematics, I will undertake a brie[illegible]
essences.[73]

The Constitution of Physical Esse[illegible] — or Composites of Form and Matter

The following text of Aris[illegible]
talking about the essence of phys[illegible]

> By form I mean the essence o[illegible]
> primary entity.[74]

An even more explicit one identifies [illegible]
distinct from the composite of [illegible]
context form means <u>substantial</u> form.

> A "part" may be a part of the [illegible]
> essence), of the composite of [illegible]
> matter, or of the matter itself.

And still another:

> [W]hen I speak of the entity [illegible]
> [which is the form][75] I mean [illegible]

All of these are from the <u>Metaphysics</u>.
<u>sics</u> the identity of essence a[illegible] form
tion concerning the degree to w[illegible]
the substantial form of a compos[illegible]
"How far then must the physicist k[illegible]
[underlining mine]

But this is too easy! If w[illegible]

things are determined, it will still remain to be seen if what is said about it can also be applied to the essence of mathematical quantities since the latter are so different from physical things. However, in the hope that it will shed some light on the makeup of the essences studied by mathematics, I will undertake a brief analysis of physical essences.[73]

The Constitution of Physical Essences — Pure Forms or Composites of Form and Matter?

The following text of Aristotle in which he is talking about the essence of physical things is very explicit.

> By form I mean the essence of each thing and its primary entity.[74]

An even more explicit one identifies essence with form as distinct from the composite of form and matter. (And in the context form means <u>substantial</u> form.)

> A "part" may be a part of the form (i.e. of the essence), of the composite of the form and the matter, or of the matter itself.[75]

And still another:

> [W]hen I speak of the entity without matter [which is the form][76] I mean the essence.[77]

All of these are from the <u>Metaphysics</u>. But even in the <u>Physics</u> the identity of essence and form is made. In a question concerning the degree to which the physicist must know the substantial form of a composite thing Aristotle asks, "How far then must the physicist know the form <u>or essence</u>?"[78] [underlining mine]

But this is too easy! If we attack our problem from

a different point of view the position of the Stagirite is not nearly so obvious. Definition, he says in many places, is of the essence.[79] Therefore, if we determine what is present in the definition of some thing we are on our way to determining of what its essence consists. It may be, of course, that a definition could include the essence and more. Nevertheless, since Aristotle tells us it is of the essence we can be certain that if something is not present in the definition it is not of the essence of what is defined. Now there are some texts which state that the definition of a physical thing is of the form alone. For example, "[D]efinition is of the universal and the form."[80] The whole context of the text is very informative:

> Another question is naturally raised, viz. what sort of parts belong to the form and what sort not to the form, but to the concrete thing. Yet if this is not plain it is not possible to define any thing; for definition is of the universal and of the form. If then it is not evident what sort of parts are of the nature of matter and what sort are not, neither will the formula of the thing be evident. In the case of things which are found to occur in specifically different materials, as a circle may exist in bronze or stone or wood, it seems plain that these, the bronze or the stone, are no part of the essence of the circle, since it is found apart from them. Of things which are not seen to exist apart, there is no reason why the same may not be true, just as if all circles that had ever been seen were of bronze; for nonetheless the bronze would be no part of the form; but it is hard to eliminate it in thought, e.g. the form of man is always found in flesh and bones and parts of this kind; are these then also parts of the form and the formula? No, they are matter . . . .[81]

Note especially the last two lines of this passage. Aristotle asks, are flesh and bones and parts of this kind part of the form and the formula? (where formula is apparently

equivalent to definition in this context).[82] And he answers, "No, they are matter!" Just as one would not define circle by including bronze so he should not include matter in his definition of man. According to this the definition, the formula, is of the substantial form alone.

There are other texts which make the same point. There is the text in Chapter 12, Book VII of the Metaphysics in which Aristotle discusses definition by means of division and concludes that "the last differentia [in a step by step division "will be the form"[83] and] will be the entity of the thing and its definition."[84] All of these texts, then, seem very explicit. Definition is of form alone; hence, the essence of physical things consists only of form.[85]

Yet there are other texts having to do with the definition of physical things in which Aristotle seems to assert just the opposite position. In the Physics, for example, he states, in opposition to those who made physical essences separate forms, that things like man, flesh, bone, for example, are defined by including matter, just as snub nose is.[86] In the Metaphysics he repeats this same point in talking about the way the science of physics defines its object:

> [P]hysics must be a theoretical science, but it will theorize about such being as admits of being moved, and about substance-as-defined for the most part only as not separable from matter. Now, we must not fail to notice the mode of being of the essence and of its definition, for, without this, inquiry is but idle. Of things defined, i.e. of "whats," some are like "snub," and some like "concave." And these differ because "snub" is bound up with matter (for what is snub is a concave nose), while concavity is independent of perceptible matter. If then all natural things are analogous to the snub in their nature—

> e.g. nose, eye, face, flesh, bone, and, in general, animal; leaf, root, bark, and, in general, plant (for none of these can be defined without reference to movement — they always have matter), it is clear how we must seek and define the "what" in the case of natural objects, . . .[87]

Clearly in these texts the definition and the essence of physical things are said to include matter. We seem, then, to be no closer to answering our question — does the essence of a physical thing consist only of form or of the form-matter composite? Some texts support the former opinion, some the latter.

Without attempting to definitively resolve this question, I would suggest that a partial solution can be found by noting the analogous use of definition, and of essence, by Aristotle. As he says, "definition and essence must be said to have more than one sense."[88] Chapters 4 and 5 of Book VII of the Metaphysics set this forth clearly. The term essence, the Stagirite says, can be said of all the categories, e.g. we can ask what whiteness is, or what running is, or what a man is, though primarily it is said of entity (ousia — in the context it is clear he is referring to the category substance) and only secondarily of the other categories.

> [E]ssence will belong, just as "what a thing is" does, primarily and in the simple sense to entity, and in a secondary way to the other categories also — not essence in the simple sense, but the essence of a quality or of a quantity.[89]

Since definition corresponds to essence, definition too will be of entity, ousia (substance), in the primary sense and of the other categories secondarily.[90]

But, and here is the main point, even in regard to the definition of entity there will be a primary and a secondary sense of definition for there is a primary and a secondary sense of entity. Entity in the primary sense refers to the substantial form of a thing: "By form I mean the essence and the primary entity of each thing";[91] entity in the secondary sense refers to the thing itself, the composite of form and matter. (The composite is a secondary or derived sense of entity because its entity comes to it from its form, which causes it to be the actual determinate thing it is.)[92]

Thus, it would seem that it is possible to have at least two kinds of definition of a thing: definition of its primary entity which would be definition of its form only; definition of its secondary or derived entity which would be definition of the composite, a definition which would include matter as well as form.[93] (Incidentally, I might note here, in case there is any question, that Aristotle seems to find no difficulty in asserting that the form alone of a composite thing can be defined. He makes statements which leave no doubt on this point.[94] See, for example, the texts quoted above, pages 46 and 47. Thomas Aquinas, as we shall see, will not agree!) It would follow, then, that essence too, since it is what is defined and is also said to correspond to entity, would have at least a twofold meaning, one including form alone and one including form and matter. This would explain the apparently contradictory statements of the

Stagirite noted above, viz. that definition and essence are sometimes said to include only form and are sometimes said to include form and matter.

Unfortunately there are difficulties in this solution. Recall that Aristotle asserts that definition and essence are said of entity. If there is a sense of definition and essence which includes both form and matter this should be referred to as entity by the Stagirite — secondary entity, no doubt, but entity nevertheless. Now the matter which the definition and essence would include would certainly not be individual matter but rather common matter. However, to my knowledge, Aristotle never mentions a sense of entity which refers to essence as including both form and common matter. In fact, the only instances of secondary entity, in regard to physical things, that he mentions refer to the individual composite and to its matter.[95] I can not find the expression common matter ever used by the Stagirite, though he does have something like it. In a few places, very few, he refers to such notions as man or animal as the compound of form and matter "taken universally."[96] (Incidentally, St. Thomas Aquinas interprets this expression as meaning form and what he calls common matter.)[97] However, Aristotle says that this compound taken universally is not entity:

> But man and horse and the terms which are thus applied to individuals but universally, are not entity but something composed of this particular formula and this particular matter treated as universal; and as regards the individual, Socrates already includes in him ultimate individual matter.[98]

If this compound is not entity, then, strictly speaking, it

could not have an essence or be defined. Perhaps Aristotle means only that it is not entity in the primary sense but rather in a derived sense. However, he never says this and, in fact, as we noted above, he mentions only three types of entity: form, the individual composite, and its matter. Is there a fourth type, form plus matter "taken universally"? He never says so. If there is not, this would mean that ultimately, in the strict sense, the only definition of a physical thing that is possible is of its form alone, and that its essence, therefore, consists only of form. This conclusion, however, seems impossible to reconcile with the texts given earlier (pages 47 and 48) which state explicitly that in the definition and the essence of physical things matter is included and with other passages where the Stagirite does speak of defining the compound of form and matter.[99]

Perhaps we can find a solution if we investigate Aristotle's notion of definition <u>by addition</u>. In a few places, he states that the definition of physical things includes matter by way of addition[100] (and the included matter would have to be matter taken universally, not individual matter). Such a definition is of form primarily and yet matter must be added for the forms of physical things do not exist separate from their matter.[101] Now some have interpreted the statement "by addition" to mean that matter, though in their definition, is not of the essence of physical things.[102] (And indeed, it is true that Aristotle

speaks of accidents as defined "by addition" of their subject and their subject is nevertheless not really part of their essence.)[103] In other words, these commentators would say that the definition of physical things includes their essence and matter, the essence being form alone. This interpretation is able to explain why, even though entity is never said of both form and matter taken universally, there is a definition which includes both. For such a definition would fulfill the requirement of being of entity and of essence inasmuch as it includes form, the matter it contains being only added on. However, this interpretation cannot explain the fact (And indeed it denies the fact.) that Aristotle asserts that physical <u>essences</u>, not just their definitions, contain matter.

On the other hand, others have interpreted the phrase "by addition" to mean that the matter in the definition of physical things is of their very essence.[104] This interpretation can explain why both definition and essence are often said to include form and matter (where the matter would be matter taken universally). Yet it does not explain why entity, to which definition and essence must correspond, is never said of form and matter taken universally. And so we are faced with the same problem. Is there for Aristotle a sense of entity and so of essence, which includes matter as well as form?

Whatever the ultimate solution is, and to attempt to fully resolve it would take us, I fear, too far from our

topic of mathematics,[105] this much is clear by now. The definition, andhence the essence of a physical thing, consists in the primary sense of form and not of form and common matter. Any other definition and essence, if there are others, and there certainly seem to be however vague their status in relation to entity, will definitely be secondary to this. It remains to be seen in what way these conclusions can apply to mathematical entities.

Application of These Notions to the Central Problem

Before attempting to make this application, however, it might be well to generally sum up the ground we have just covered, and repeat our reason for covering it in the first place. The foregoing analysis was undertaken in order to attempt to give some answer to the question — does the essence of a thing for Aristotle consist only of the form or of form and common matter? This question was raised in the hope that an answer to it would be of aid in determining whether the essence of mathematical quantities for Aristotle consists only of form (quantitative form), as some texts indicate, or of form and common intelligible matter. Our conclusion of the immediately preceding analysis of essence and definition was that in the case of physical things there is definitely a sense in which essence consists of form only. Furthermore, this notion of essence is more primary than any other sense of essence, if there are others, which might include form and common matter. Can this conclusion be extended to mathematical quantities such that their essence in the

primary sense at least could be said to consist of quantitative form only?

Some of the texts cited earlier in reference to the line and the circle would indicate that it can. A text given previously in reference to a line and its matter is worth repeating, adding to it a few more lines at this time. Note, incidentally, that the passage refers in the same breath to both a physical and a mathematical object. No distinction is made between the two, implying that what is true in the case of one will hold true in the case of the other.

> [E]ven if the line when divided passes away into its halves, or the man into bones and muscles and flesh, it does not follow that they are composed of these as parts of their essence, but rather as matter; and these are parts of the concrete thing, but not also of the form, i.e. of that to which the formula refers; wherefore they are not present in the formula. In one kind of formula, then, the formula of such parts will be present, but in another it must not be present, where the formula does not refer to the concrete object.[106]

Here the distinction is clearly made between two kinds of formula: one of the form only (and he calls this the essence), and the other apparently referring to the concrete object and containing material parts also. The parallelism between these two formulae and the possibly two kinds of definition of physical things referred to above is obvious. The two notions of essence which <u>could</u> correspond to these formulae (definition), one containing form only and one also including matter, is also obvious. (I say <u>could</u> correspond because it does not necessarily follow that everything contained in the definition is of the essence. Recall Aristotle's

"definition by addition" discussed above. The definition must contain the essence but it could contain more.)

Later in the same chapter this same distinction between two kinds of formula seems to be applied to the formula of the circle and that of the right angle. Both of these objects are discussed along with physical things without any indication from Aristotle that what applies in the case of one would not apply in the other. Apparently his remarks apply to both kinds of things.[107]

However, the texts are few, and we have already seen Aristotle point out some of the significant differences between quantified physical things and mathematical quantities. Before we can be certain that the notion of essence which applies to physical things is true also of mathematicals, we should investigate the sense in which mathematical quantities can be said to be entities, for as we saw above (page 48) essence (and definition) is said in reference to entity.

Entity in the primary sense was said to belong to the first category, that which is in itself, and not in another. Are mathematical quantities entities in this primary sense as quantified physical things are? Certainly they are not quantified physical substances. Recall, mathematicals are nonsensible, not always three-dimensional, not in place, and immobile. Furthermore, inasmuch as they are immobile, mathematicals must to some extent lack matter which is present in all physical things as their principle of motion.[108] No wonder then that Aristotle considers

"strange" the Pythagorean position that physical things are composed of mathematicals.[109] How can the three-dimensional be composed of that which lacks three dimensions; the perceptible of the imperceptible; that which is in place of that which is not; that which is mobile and fully material of that which is immobile and to some extent immaterial?

Nevertheless, even though the mathematical quantities are not quantified physical substances neither do they exist in themselves as substances separate from physical things. Nor do they exist separate from physical things in a second sense, viz. as in some way innate in man's mind. The following are just a few of the texts in which the Stagirite affirms that the objects of mathematics are not separate from natural things.

> Nor does the science we are now seeking [viz. first Philosophy] treat of the objects of mathematics; for none of them can exist separately.[110]

and another

> [M]athematics is theoretical, and is a science that deals with things that are at rest, but its subjects cannot exist apart.[111]

Again,

> [I]t is evident that the objects of mathematics do not exist apart . . . .[112]

His position is clear. The objects of mathematics are <u>not</u> substances which exist apart from physical things. In other words, mathematics <u>in some way</u> does deal with physical things and their attributes. Note the following texts,

> Obviously physical bodies contain surfaces and volumes, lines, and points, and these are the subject-matter of mathematics.[113]

> But if we are to suppose lines or what immediately follows them (I mean the primary surfaces) to be principles, these are at all events not separate substances but are sections and divisions, the one of surfaces, the other of bodies (as points are of lines); they are also extremities or limits of the same things; but all of them subsist in other things, and no one of them is separable.[114]

Nor is this surprising for, as we pointed out as early as Chapter I, quantity for Aristotle is an accident. Now, he says, "none of the categories other than substance can exist apart."[115] As an accident quantity is an attribute which exists only in physical substances. Quantity only is in something quantified. The Stagirite asserts,

> [E]verything except primary substance is either predicated of primary substance, or is present in them, and if these last did not exist, it would be impossible for anything else to exist.[116]

To repeat, then, mathematical quantities are not entities in the primary sense, i.e. existing in themselves, apart from physical things. Instead, "none of them can exist separately," they "cannot exist apart." This can only mean that for Aristotle mathematics does in some way study the attributes of physical things.

But doesn't this contradict an earlier statement which said that the mathematician is dealing "not with the things in this world"? (page 20) Indeed, we seem to find Aristotle holding on the one hand that mathematical quantities do not exist separate from physical things, and on the other hand asserting that it is <u>not</u> such things that mathematics studies and violently opposing the Pythagoreans who held that physical things contain mathematical entities.

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The Stagirite himself was well aware of this apparent contradiction in his thought as the following passage shows:

> It has been sufficiently pointed out that the objects of mathematics. . . are not prior to sensibles in being, . . . and that they cannot exist somewhere apart. But since it was not possible for them to exist in sensible things either, it is plain that they either do not exist at all or exist in a special sense and therefore do not "exist" without qualification. For "exist" has many senses.[117]

Clearly in order to solve this dilemma we must determine in what sense mathematicals do exist. This really amounts to asking the same question we asked earlier about the kind of entity they are. We asked if the quantities which are the objects of mathematics existed as entities in the primary sense of the term, viz. that which applies to the first category, and if not what kind of entities they are. We have answered this partially. Mathematical quantities for Aristotle clearly are not entities existing in themselves as substances separate from physical things. Yet it is hard to see how they can exist in such things either. In fact it seems that he holds that they both do and do not exist in physical things.

The solution to this dilemma will come by analyzing the second type of existence separate from physical things which we mentioned above. Recall, we said that according to Aristotle mathematical quantities do not exist separate from physical things: (1) as substances, (2) as in some way innate in man's mind. In regard to the second, it is definitely true that Aristotle holds that mathematicals are not innate. His general position that all man's knowledge

is through the senses is well known.[118] And he indicates that the notions of mathematics are gained through induction based on sensation.[119] However, though they are not innate there is a sense in which it is true to say that the Stagirite holds that mathematicals do exist only in man's mind, and in this sense are separate from physical things. What follows will develop this point.

The Key to the Nature, Entity, and Existence of Mathematical Essences — Abstraction

Since all of man's knowledge comes through the senses, it could be asked what aspects of sensible things are most likely to be the basis of mathematical objects. The obvious answer is the quantitative features of such things. In his list of sensible features which are perceived by more than one sense Aristotle includes number (plurality), figure, and magnitude, the first and last of which are clearly quantitative attributes, the second a quality of quantity.[120] Sensation of these common sensible attributes, then, would apparently be the basis for the notions of mathematics. Indeed, in a text of his Nicomachean Ethics, Aristotle does indicate just that, viz. that mathematical objects are in a way perceived as common sensibles.[121]

However, to say that the quantitative features of things which are common sensibles are the basis for the notions of mathematics does not at all mean that mathematical quantities can be simply identified with common sensibles.[122] Mathematical quantities and the quantities of physical things cannot be the same for the reasons we saw earlier. Mathematical

quantities are not sensible. Physical quantities which are common sensibles are. Physical quantities are in motion and place and are three-dimensional inasmuch as they are features of physical bodies which are in motion and place and are three-dimensional; mathematical quantities are not. The common sensibles which are physical quantities are material inasmuch as they are attributes of material things; mathematical quantities are to some extent immaterial.

Nevertheless, Aristotle insists that it is the quantitative attributes of natural things, the common sensibles, which the mathematician studies. However, and this is the key point, he emphasizes that it is not <u>as attributes of natural things</u> that these quantitative features are studied.

> Obviously physical bodies contain surfaces and volumes, lines and points, and these are the subject-matter of mathematics . . . . Now the mathematician, though he too treats of these things, nevertheless does not treat of them as the limits of a physical body, nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable from motion . . . .[123]

In other words, the mathematician separates in thought the quantities of things from the things whose quantity they are.[124] He treats quantities only, as though they existed independently, and ignores the things of which they are the quantity, even though as a matter of fact the quantities do not exist separate from such things.[125]

> [T]he mind when it is thinking the objects of Mathematics thinks as separate elements which do not exist separate.[126]

The Stagirite explains that one aspect of a thing can be studied without taking all of its other properties into

account, and it is not necessary that this aspect actually exist apart from all the other properties. For example, a moving thing can be treated insofar as it is moving without considering the kind of thing it is nor any of its accidental features. Yet this does not mean that there actually is a mobile apart from these qualities. In like manner, simply the quantity of a physical thing can be studied, all its other characteristics being disregarded, even though the quantitative does not exist separate from the thing with all its other features. This is precisely what the mathematician does according to Aristotle. He does, in a way, treat of physical things but only insofar as they are quantitative, i.e. insofar as they have parts distinct from parts (in the sense of magnitude or plurality) and, therefore, are divisible in some way into quantitative parts, and he leaves aside all qualitative and other features.

> [C]learly it is possible that there should also be both propositions and demonstrations about sensible magnitudes, not however qua sensible but qua possessed of certain definite qualities. For as there are many propositions about things merely considered as in motion, apart from what each such thing is and from their accidents, and as it is not therefore necessary that there should be either a mobile separate from sensibles, or a distinct mobile entity in the sensibles, so too in the case of mobiles there will be propositions and sciences, which treat them however not qua mobile but only qua bodies, or again only qua planes, or only qua lines, or qua divisibles, or qua indivisibles having position, or only qua indivisibles. Thus since it is true to say without qualification that not only things which are separable but also things which are inseparable exist (for instance, that mobiles exist), it is true also to say without qualification that the objects of mathematics exist, and with the character ascribed to them by mathematicians . . . . [I]f its subjects happen to be sensible, though it does not treat them qua sensible,

> the mathematical sciences will not for that reason be sciences of sensibles — nor, on the other hand, of other things separate from sensibles.[127]

Mathematics, then, does in a way deal with the attributes of sensible physical things; however, it is not a science of sensible physical things for it does not treat such things qua physical or qua sensible but only qua quantitative, viz. in terms of magnitude and/or quantitative plurality only. To take another example from Aristotle, the geometer, the arithmetician, and the physicist can all in a way consider man. However, man qua man is not an object for the two mathematical sciences. The geometer treats man only as a continuous quantity, e.g. as a three-dimensional solid; the arithmetician treats man only as a unit and hence as expressible in number.[128] Neither treats man as a physical entity; furthermore, neither treats its quantitative attributes as attributes of a man.

That last remark should be stressed. For Aristotle mathematics does treat the quantities of physical things (Indeed, there are no other quantities!) but not as quantities of physical things. This, I think, is the best way to express his position. To simply say that the mathematician studies "physical things qua quantified" though true tends to be misleading. For this statement seems to imply that the quantities are studied as attributes of physical things, and this is certainly not the case with arithmetic and geometry as we have seen. (There are mathematical sciences which study the quantitative attributes of physical things

and we will discuss them in Chapter IV, but we are not talking about such sciences here.) As the Stagirite said, the mathematician separates in thought the quantity from the physical thing. Thus, to repeat, the best way, in my opinion, to state Aristotle's position is this — mathematics treats the quantities of physical things but not as quantities of physical things.

The mental act whereby the mathematician considers the quantitative attributes of physical things separate from the things and all their other features in which they actually exist is called abstraction by Aristotle.[129] It is the act whereby the mathematician subtracts from physical things all of their other than quantitative attributes and considers only their quantitative ones.[130]

> . . . the mathematician investigates abstractions (for before beginning his investigation he strips off all the sensible qualities, e.g. weight and lightness, hardness and its contrary, and also heat and cold and the other sensible contrarities, and leaves only the quantitative and continuous, sometimes in one, sometimes in two, sometimes in three dimensions, and the attributes of these qua quantitative and continuous, and does not consider them in any other respect, ...[131]

The act of abstraction, therefore, confers on the quantitative features, the common sensibles, their specific nature as mathematicals. By that I mean that it is precisely because they are considered separate from sensible physical things and their other features that the quantities of these things are nonsensible, immobile, free from place, less than three-dimensional, and immaterial to some degree. And it is only in thought that they exist that way. The

mathematical quantities exist as such only by and in the act of abstraction which considers the quantities of physical things separate from these things. Aristotle states, "[T]he objects of mathematics exist by abstraction."[132] This is what I meant by saying above (page 59) that according to the Stagirite mathematicals do exist only in man's mind.[133] (It should also be clearer now why notions like generation and corruption apply to mathematicals only in an analogous sense. These "substantial changes" take place only in the mind of the mathematician combining and separating abstracted quantities.)

Though it may not be immediately apparent, we have now answered our question as to the existence of mathematicals. We have resolved what seemed to be a contradiction in the Stagirite's thought regarding their existence both in, and not in, physical things. Insofar as mathematical quantities are in a way the quantitative attributes of physical things, for it is these features, the common sensibles, which are separated by abstraction from things, the objects of mathematics exist outside the mind in physical things. Thus Aristotle can affirm, ". . . it is evident that the objects of mathematics do not exist apart." (Cited on page 56) On the other hand, as mathematicals (i.e. as having the peculiar nature that mathematical quantities do) the quantities which are the objects of mathematics actually exist only when they are known, only when they have actually been separated in thought from physical things and their attrib-

utes by the act of abstraction. Thus Aristotle can also say that it is impossible for mathematical objects to exist in sensible things. (page 58)

It is, therefore, true to say that for Aristotle mathematicals both exist and do not exist in sensible things. The apparent contradiction is resolved by noting that a different sense of "exist" is meant in each case. As the quantities of physical things which are able to be separated in thought, mathematicals as such exist potentially in things. As mathematical quantities, i.e. as nonsensible, immobile, etc., these quantities actually exist only in the mental act which abstracts them from physical things. The Stagirite himself refers to these two senses of "exist" when speaking of the objects of geometry, and his remarks can be extended to all mathematicals.

> Thus, then, geometers speak correctly; they talk about existing things, and their subjects do exist; for being has two forms — it exists not only in complete reality [i.e. actually] but also materially [i.e. potentially].[134]

This solution will also enable us to answer our question regarding the type of entities that mathematical quantities are for Aristotle. We have seen earlier that his position is that they do not exist as entities in themselves, as substances, separate from physical things. We have now seen that they exist in physical things potentially, insofar as they are identical with the quantitative features of such things which are able to be separated in thought. Now as identical in the aforesaid way with a thing's quantitative features mathematical entities are certainly not entities

in the primary sense, but are secondary entities, accidents, dependent upon primary entities, substances; they can be only in another. Quantities do not exist; quantified physical things do.

Furthermore, inasmuch as mathematical quantities as such exist actually not in themselves but only in the mental act of the mathematician, they are again entities only in a secondary sense. Some modern commentators have said that for Aristotle actual mathematical quantities are "beings of reason."[135] Though the phrase "beings of reason" is not Aristotle's, I think it can be properly used to designate the fact that as far as their actual existence is concerned mathematicals as such are only in the mind of the mathematician. Of course, in calling them "beings of reason" I do not mean to imply that mathematical quantities are pure fictions concocted by the mind, nor even that their real existence would be self-contradictory (as the real existence of the logical intention of species is). As a matter of fact, the mathematical quantity existing with its peculiar properties only in the mind of the mathematician is real quantity, real quantity however lacking all other features of real things. As Aristotle says the mathematician considers real quantity not qua real. In both their potential and their actual existence then, mathematical quantities are entities only in a secondary sense, though the sense is different in each case.[136]

Partial Solution to Central Problem — the Presence of Substance in Abstracted Quantity

The above conclusion will help us solve our major

problem regarding the essence of mathematical quantities. Recall that we have asked if these essences, the essences treated by the science of mathematics, consist only of quantitative form or of form and common intelligible matter. We found that in the case of physical things, which are entities in the primary sense, there definitely is a sense in which their essence consists of form only for form is the primary entity and the source of the composite entity. We noted also that there may be a secondary meaning of essence which includes both form and common matter, though Aristotle leaves the status of this, i.e. whether or not it is entity, very vague. (pages 50 and 51)

Now we have shown that Aristotle holds that mathematical quantities are entities only in a secondary sense, only as inhering in primary entity. The only real nonmental existence that they have is as accidents of physical things. Certainly as entities of this type quantities can be said to be essences and have a definition only in a secondary sense, for essence and definition are directly proportionate to the type of entity, as we have seen. (page 48) And this is a crucial point, for Aristotle states that the way in which any secondary entity, any accident, any of the categories other than primary entity, is definable is as depending upon and hence including primary entity. An accident can be only by inhering in a primary entity as a substrate and so can be defined only by including its substrate in its definition.[137] He calls the definition of accidents a definition by addition.

> Clearly, then, only substance is definable. For if the other categories are definable, it must be by addition of a determinate, e.g. the qualitative is defined thus . . . .[138]

To say it another way, mathematical forms are accidental forms of quantity and therefore are not intelligible purely in themselves but only with reference to a substrate in which they inhere. The reason for this is that they are not entities like substantial forms. These latter according to Aristotle are not so dependent upon the composite, for the composite derives its entity from them; hence they can be defined separate from the composite and said to be essences themselves. (This was discussed above, page 49.) The accidental forms, on the other hand, are, according to the Stagirite, completely dependent upon the physical composite for their being; they are entities only by reference to it, the primary entity. But definition, he says, is of entity and has as many senses as entity does. If accidental forms are entities only by reference to primary entity as their substrate then they must be defined with reference to a substrate. Therefore, in the case of mathematical quantities, which are nothing other than abstracted accidental quantities, their definition must include a substrate as well as the quantitative form. Thus, Aristotle's doctrine would apparently have to be that the mathematician cannot really completely separate the quantitative forms of physical things from their material substance.[139]

And yet, strangely enough, I know of no place where he makes this conclusion. I know of no place where he says

in so many words that the abstracted object of mathematics is not simply quantity but is actually quantified material substance. True, he never denies it. But he never explicitly affirms it either. He does say that numbers must be numbers of something, at least of units.[140] He does stress, as we have seen, that the mathematician deals with real quantity and that such quantities are only accidents, and he even speaks at times as if the mathematician studies physical things qua quantified (cf. passages referred to in footnotes 127 and 128) — a statement which, though misleading, clearly includes a substrate of the quantities. However, in other places, as we have also seen, in referring to the mental status of abstracted quantity he speaks as if the mathematician studies the quantity alone, which quantity he has separated in thought from the thing. For example, he asserts that the mathematician by his stripping off leaves only the quantitative features (cf. Aristotle's text quoted on page 63), and again he speaks of him separating only the quantitative features from things (cf. Aristotle's text quoted on page 60, footnote 123). Take the following text for example which implies that the mathematician separates quantity from any and every substrate:

> The so-called abstract objects the mind thinks just as if one had thought of the snub-nosed not as snub-nosed but as hollow, one would have thought of an actuality without the flesh in which it is embodied; it is thus that the mind when it is thinking the objects of mathematics thinks as separate, elements which do not exist separate.[141] [underlining mine]

Thus, it seems that sometimes in his desire to stress the fact that the mathematician does not consider quantity as

the quantity of physical things, Aristotle has gone so far as to omit reference to any subject of the abstracted quantity. As I said just above, I know of no place where he unequivocally enunciates the principle that mathematical quantities must include material substance as their substrate. Be this as it may, his doctrine concerning the nature of accidents and their definition is perfectly clear. No accident, and that includes quantity, can be separated in being or in thought from substance. Why doesn't he affirm this in reference to mathematical quantity? The answer seems to be that we have here another instance (The first was noted in Chapter I, pages 6 and 7.) where the Stagirite simply neglected to explicitly apply a general philosophical doctrine to the special area of the philosophy of mathematics. Nevertheless, what its application must be seems beyond question.

In what sense, then, would a physical substance, the substrate of its accident of quantity, enter into the intelligibility of this quantity when it is abstracted? We have seen Aristotle point out that the mathematician leaves aside all the qualitative features of a thing. He also apparently holds that the mathematician leaves aside the specific nature of the thing from which he abstracts quantity. Recall that he states that man qua man is not included in the mathematicians' object, only qua solid or some other quantity. Recall too his even stronger statements to the effect that only quantity is mentally separated from the physical thing. It seems then, that substance would be present in the intell-

igibility of abstracted quantity only as an indeterminate unspecified material substrate of the quantity. Only in the very general sense that quantity must be grasped as the quantity of something would physical substance remain in the abstracted mathematical object — though we must hasten to add that even this extremely minimal presence of substance in the mathematical object is not explicitly mentioned by Aristotle.

Probably the main reason for the Stagirite's neglect of this point is the fact that he gives practically no explanation of why mathematical abstraction is possible in the first place. He never clearly says why the mathematician can leave aside some features of a physical thing while he studies others nor does he state what cannot be left aside. He never gives reasons why the mathematician is free to ignore the nonquantitative accidental features of a physical thing, nor why he can neglect the specific nature of the quantified physical substance which is the basis of his abstraction. He never sets forth a general principle stating what can and what cannot be set aside by abstraction and why.[142] If he had enunciated some principle on this point, it would have determined what must be included in the abstracted mathematical object; it would have, I think, made it clear that substance cannot be totally omitted. It is true that Aristotle does state emphatically that no error is involved in the mathematician's abstraction:

> [I]f we suppose attributes separated from their fellow attributes and make any inquiry concerning

them as such, we shall not for this reason be in error . . . .[143]

But why not? Why can quantity be separated in thought without error from all the actions, passions, qualities, etc. of a physical thing and even apparently from its substantial form when it does not exist separate? No reason is given. As far as the nonquantitative accidents are concerned, it must be that somehow quantity does not depend on these other secondary entities for its entity as it does depend on substance. If it depended on other accidents it would have to include them in its notion just as it has to include substance. Yet I know of no place where the Stagirite enunciates this doctrine. To put it another way, I know of no place where he sets forth the ontological basis for the abstraction of quantity from all other accidents.[144] He simply says that abstraction can be done without error.

Concerning the mathematician's abstraction from the specific substantial form Aristotle is only slightly more explicit. He does distinguish between attributes which in their explanation demand a determinate subject and those which do not. The former are attributes "in which is involved either the <u>formula</u> or the <u>name</u> of the subject of the particular attribute, and which cannot be explained without this . . . ."[145] The latter are attributes whose explanation, equivalent to definition in the context, apparently does not involve the formula or name of their subject. However, like other doctrines this distinction is never applied to mathematical quantities by the Stagirite. Since such quantities are apparently ab-

stracted from a particular specific subject they must be attributes whose definition does not demand the inclusion of a determinate subject. (To illustrate, the definition of a mathematical triangle does not demand, indeed it cannot include, the notion of iron though a particular physical triangle may be of iron; likewise the definition of seven does not include the notion men though seven physical things may be men.) However, this conclusion is mine; Aristotle does not draw it. As I said he never applies his distinction between the two types of definitions of accidents to mathematical quantities.

Thus in the case of both the accidental and the substantial features of a thing practically no reason is given by the Stagirite why the mathematician can, without error, abstract from most of these features while concentrating on quantity. And, an even more crucial omission is his failure to set forth a general principle stating what aspects of a physical thing can be abstracted from other aspects and what cannot, and why. As I suggested above, it is the lack of this principle which accounts for Aristotle's failure to mention the presence of material substance in the object of mathematics.

Nevertheless, even though the Stagirite doesn't say so, it seems clear according to his own principles that the mathematician, as a result of his act of abstraction, ends up not with quantity alone — it can neither be nor be known alone (This he does say.), — but with a quantified but

otherwise indeterminate material substrate. The physical substance is present in the objects of mathematics in its role, and only in this way, as a substratum for the abstracted accident of quantity. Since this material substrate is indeterminate except for its quantitative form, why not call it matter? After all matter primarily means that which is totally indeterminate.[146] To distinguish this matter, which is present in entities which actually exist only mentally, from the matter found in sensibly existing things, call it intelligible matter.[147] The mathematician after abstraction, therefore, ends up not with pure quantitative form but with a quantified but otherwise indeterminate material substrate, i.e. quantified intelligible matter.

Final Solution to Central Problem — the Two Kinds of Intelligible Matter

Is this ultimately what Aristotle means by intelligible matter — a corporeal substance functioning as the substrate of abstracted quantity? He never says so,[148] but then he actually uses the phrase "intelligible matter" only three times in all of his works;[149] and, as I've continually pointed out, he never even mentions the presence of substance in the objects of mathematics. However, the Stagirite does refer to the intelligible matter of mathematicals as their substrate, for he says that this matter accounts for a mathematical not being simply a bare form but a "this" (where "this" in the context means a concrete

individual).

> [E]ven some things which are not perceptible must have matter; indeed there is some matter in everything which is not an essence and a bare form but a "this" . . . for while one kind of matter is perceptible, there is another which is intelligible.[150]

He also speaks of this matter as accounting for different individual mathematicals of the same kind. He points out that,

> [T]he objects of mathematics, while they differ from the things of this world in some other respect, differ not at all in that there are many of the same kind.[151]

For example, there can be many circles, many equilateral triangles, many number 2's. Since matter, he says, is the principle of individuation,[152] that which accounts for the same quantitative form (e.g. circularity, triangularity, twoness) being found in different individuals would be intelligible matter.[153]

It is clear, then, that the intelligible matter of mathematics does for the Stagirite function as a substrate of the quantitative forms for it accounts for mathematical quantities not being pure forms but individuals, "thises."[154] It is also very clear that he never states the identification of this substrate, this intelligible matter, with corporeal substance subject to quantity. Nevertheless, as we have tried to show above, it does seem that the two are identical.

If we are correct in maintaining this identity we can now proceed to answer the main question posed earlier concerning the constitution of the essence of mathematical

quantities. Is this intelligible matter part of that essence? The answer is no! True, substance as the substrate, the intelligible matter, of quantity must be included in the definition of quantity and hence of mathematical quantity, it must be part of the total abstracted object; but this does not mean that it is part of the very essence of mathematical quantity. As we have noted earlier, Aristotle calls definitions of accidents definitions by addition. And he points out that that which is added to the accident in its definition is not actually part of the essence of that accident. To put it simply, substance is not quantity; therefore, it is not part of quantity's essence.[155] Let us follow the Stagirite in his discussion of this point. He asks, is an accidental term, like white, the same as its essence? In answer he first distinguishes two meanings of white,

> . . . both that to which the accidental quality belongs, and the accidental quality, are white.

And then he reasons,

> [I]n a sense the accident and its essence are the same and in a sense they are not; for the essence of white is not the same as the man or the white man, but it is the same as the attribute white.[156]

Clearly in this passage the essence of the accident (white) is identified only with the accident itself (attribute white) not with its substrate (man) nor with the accident plus its substrate (white man). Applying this same distinction to quantity we must conclude that for Aristotle the essence of quantity is the same as the attribute quantity but not the

same as its substrate or as the accident plus its substrate. Thus, the material substance included by addition in the definition of quantity, and so, in the definition of mathematical quantity, which is nothing less than abstracted real quantity, is not part of the essence of such quantities. It follows that intelligible matter as the substrate, as identical to the corporeal substance functioning as the subject of the mathematical quantity, is added to the essence of this quantity in its definition and is not part of that essence. It is part of the total abstracted object, part of the definition of abstracted quantity, but not part of the quantitative essence itself.

But what about intelligible matter in the sense of the parts of mathematicals? We have seen that the Stagirite refers to the parts into which mathematical quantities are divisible and out of which they are composed as their matter; e.g. the segments of a line and a circle, the units of a number. Indeed, he attributes intelligible matter to such parts at least as often, if not more often, than he predicates it of the individuating substrate of the mathematical quantitative forms. Must intelligible matter in the sense of these parts be included in the definition of quantity, and if so does this make it part of the essence? Or can the mathematician abstract from, i.e. leave aside, these parts in obtaining the essence of his object? Unfortunately once again Aristotle doesn't say.

As a matter of fact, we noted upon examining those

passages in which he refers to this matter (pages 38 to 43), that it seems to be relegated only to individual mathematicals. Except for one reference to a mathematical demonstration through material cause, he seemed to refer to this matter only as individual, never as common matter, therefore never as present in the essence of mathematical quantities. And yet it would seem that such matter, the parts of mathematicals, would have to be part of the essence of quantity. It would seem that the mathematician could not abstract from them. As we showed back in Chapter I it is Aristotle's teaching that it is the very nature of quantity to have parts distinct from parts. He describes the quantitative as that which is divisible into parts, and we showed that the actual attribute of quantity allowing for this divisibility is an actual unity of distinguishable quantitative parts. How then could such parts, such matter, be left out of its definition or its essence? (To take simple examples, how could a triangle be what it is without its three lines, or a number without its units?) Furthermore, we saw just above that the Stagirite teaches that the essence of an accident is the very accident itself and no more. But as we noted in Chapter I the accident quantity for him is actually nothing more nor less than a unity of many distinguishable quantitative parts. In distinguishing quantity from quality we pointed out that for him quantity is not that which accounts for things having certain habits or dispositions, capacities or incapacities, affective quali-

ties, affections, figures, or essential differences. Quantity rather is that which accounts for things having a certain magnitude or being a quantitative plurality. Quantity is that accident which consists of a unity of many parts and which, therefore, accounts for the extension of a thing or for the number of quantitative units. Without such parts there simply is no quantity. All of these considerations then seem to demand that intelligible matter, meaning the parts of the mathematical quantities, be included in the definition and essence of these quantities for Aristotle.[157] The mathematician simply cannot "strip them away" to arrive at his quantitative essence or his definition. Yet as we have pointed out earlier the Stagirite never refers to a common intelligible matter present in mathematical essences or definitions. Indeed, as we noted in Chapter I he never says that quantity by its very nature or essence has a relation to matter, though such a relation seems to be necessary. Furthermore, his texts indicate, though they do not explicitly say so, that he believes that intelligible matter is found only in individual mathematicals. Once again we could wish that the Stagirite had set forth some principle stating what the mathematician in his concentration on quantity can and cannot abstract from and why.

Conclusion

What then can be concluded concerning Aristotle's doctrine about the essence of mathematical quantities? Is this essence pure form or form and common intelligible mat-

ter? It appears that we have no other choice than to admit that the Stagirite seems never to have conceived of the notion of a common intelligible matter present in the essence and in the definition of mathematical quantities. He apparently limits intelligible matter to individual mathematicals as both their parts and their individuating substrate. On the other hand, we must acknowledge that it seems to follow from principles he does enunciate that intelligible matter would have to be present in the definition and also in the essence of mathematicals. For one thing, his doctrine that quantity demands a substance even in being known leads to the inevitable conclusion that this substance (which seems to be identical to the intelligible matter which he cites as the individuating substrate of mathematical forms), is part of the total mathematical object. Yet this intelligible matter would not be part of the very essence of mathematical quantities as we have seen. However, intelligible matter in the other sense, meaning the parts of mathematical quantities, would seem to be part of the essence as well as the definition. The very nature of quantity as Aristotle explains it involves a relation to this matter. His teaching that quantity is by nature a composition of parts, as well as his views on the essence of accidents, seem to demand that he say that these parts, this intelligible matter, be present in the essence of mathematical quantities as well as in their definition. In one sense then, where it refers to the parts of quantity, intelligible matter would be part

of the essence; in the other sense, where it means the substrate of the accident quantity, it would be part of the total abstracted object, part of the definition of mathematical quantity, but not part of its essence. Of course, in drawing such conclusions, we are in the awkward position of saying what it seems Aristotle, on the basis of his own principles, should have said, and, yet as a matter of fact, did not say.

What can account for the Stagirite's omissions in this matter? Why didn't he apply his doctrines on the nature of the accident quantity to the area of mathematics? Furthermore, why wasn't he clearer as to why the abstraction of quantity is possible without error, or more generally as to what can and what cannot be left aside by the act of abstraction without involving the mathematician in falsity? Indeed, as we have said, this last question is more fundamental than the others. For if Aristotle had clearly enunciated some principle which stated why certain entities can and certain entities cannot be abstracted from others by the mathematician, he would have thereby made clear what is and what is not part of his total abstracted object, and this would have helped determine the ingredients of the essence of abstracted quantity. But he sets forth no such principle. It is true he unequivocally asserts that accidents must be known in relation to substance, but he never draws from this a general principle which would apply to all abstraction. In like manner, he does give rea-

sons why some accidents can be known apart from a determinate subject and some cannot; but he does not develop this point into a general norm applicable even to the mathematical abstraction of quantity.

Perhaps these omissions can be explained by noting what seems to be the Stagirite's general lack of interest in mathematics (the reasons for which we will see later). After all, a man cannot say everything about everything, and he will certainly neglect areas in which he is not particularly interested. (I think too that his overwhelming stress on the form of physical things as their primary entity and essence may have led him to unduly minimize the presence of common intelligible matter in the objects of mathematics and in the mathematical essences.) Whatever the reasons be, there is definitely a gap in Aristotle's philosophy of mathematics in this area, a gap which to a great extent, as we shall see, will be filled in by at least one of his disciples, St. Thomas Aquinas.

FOOTNOTES

[1]Metaphysics, XI, 1, 1059b 8-11.

[2]The following are some of the places where Aristotle calls mathematicals nonsensible: De Sensu et Sensibili, translated by J. I. Beare, Vol. III, The Works of Aristotle translated into English under the editorship of W. D. Ross (Oxford, 1954), 6, 445b 13-15; Metaphysics, I, 8, 989b 29-32; XI, 3, 1061a 29-36.

A. Mansion who identifies mathematical quantities with common sensibles states that Aristotle should not have said that the objects of mathematics are nonsensible. (Introduction à la Physique Aristotélicienne (2d ed.; Louvain, 1946), p. 169; "La physique aristotélicienne et la philosophie," Revue Néoscholastique de Philosophie, XXXIX (1936), 14-15) In my opinion, M. Mansion is wrong in this identification and, indeed, it vitiates much of his understanding of the Stagirite's philosophy of mathematics. See also footnotes 27 and 122 of this chapter.

[3]Just a glance through Euclid's Elements will reveal that he is treating of figures and numbers lacking sensible qualities. A check of Heath's detailed work, . . . Greek Mathematics, will show that the objects studied in geometry and arithmetic (as distinct from those treated by geodesy [geometry applied to physical magnitudes] and logistics [arithmetic applied to physical pluralities], I, pp. 13-16) are with such features. This is not to deny that such entities have sensible counterparts and indeed result ultimately from our knowledge of sensible quantities. We shall discuss this later.

[4]Metaphysics, III, 4, 1001b 11-12. To have "being in every dimension" is to be three-dimensional for that is all the dimensions there are, as Aristotle states in the De Caelo, translated by J. L. Stocks in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), I, 1, 268a 9-12.

[5]Metaphysics, XIII, 8, 1083b 13-17.

[6]De Caelo, I, 1, 268a 8-9.

[7]Metaphysics, XI, 7, 1064a 31-32.

[8]Metaphysics, I, 8, 989b 31-33.

[9]*Physics*, II, 1, 192b 22-23.

[10]*Physics*, V, 1 and 2 contains his detailed analysis of the kinds of motion. At the beginning of Chapter 2, he states categorically,

"In respect of substance there is no motion."

On this point, see also Vincent Smith, *The General Science of Nature* (Milwaukee, 1958), pp. 333-338 and Ross, *Aristotle*, pp. 81-83.

[11]*Physics*, VII, 3.

[12]*Metaphysics*, V, 14, 1020a 35 - b2.

[13]It is true that in the *Metaphysics* (V, 14, 1020b 1 ff.) the Stagirite does speak of the "differentia of the essence," both in reference to physical substances and mathematical entities (both figures and numbers) as *qualities*. In the case of geometrical figures their differentia is of course their shape and shape he classifies as the fourth species of quality (*Categories*, 8, 10a 11 ff.). For Aristotle each of the categories refers to an *irreducible* attribute of physical being, for he explicitly calls them the different kinds *of being* (*Metaphysics*, V, 7, 1017a 24 and VII, 3, 1029a 21) and spends a great deal of time, especially in the *Categories*, distinguishing one from another in terms of the aspects of being it represents; indeed, the fact that a thing can change in quality and not in quantity shows they are different real attributes of the thing. Therefore, he can not literally mean that the differentia either of a substance or of a mathematical quantity is found in the category of quality. Certainly the differentia which accounts for a substance being a particular *kind of substance* would have to be in the category substance. If it were a quality it could not make something a kind of substance. And in the *Categories* Aristotle himself does refer to the specific difference as a "substantial" quality and distinguishes it from a pure quality like white. (5, 3b 18-21; also see *Metaphysics*, XI, 12, 1068b 16-19.) (See Owens' discussion of this, *. . . Christian Metaphysics*, pp. 170-171.) In like manner, the differentia accounting for a mathematical quantity being a particular kind of quantity, e.g. the shape making a figure a triangle rather than a square, that which makes a number this number rather than that, must itself be in the category quantity. If it were a quality, it could not make a mathematical a specific kind of quantity; and Aristotle does assert that mathematical quantities differ *in quantity*:

"[N]umber *qua* number differs in *quantity*."
(*Metaphysics*, XIII, 7, 1083a 4)

What then does Aristotle mean by referring to the essential differentia as a quality? The only sense I can

make of this is that he means quality only in a very general sense, as that which accounts for something being of such and such a determination.

[14]*Metaphysics*, V, 13, 1020b 2-8.

[15]*Metaphysics*, VIII, 3, 1043b 37 - 1044a 1.

[16]*Metaphysics*, X, 1, 1053a 30.

[17]*Metaphysics*, XIV, 4, 1092a 20.

[18]In Chapter III, we will discuss at length the question of what exactly is the power by which the mathematician grasps his objects. I use the expression "in thought" here simply because that is what Aristotle uses.

[19]*Physics*, IV, 1, 208b 23-25, translated by Richard Hope with an Analytical Index of Technical Terms (Lincoln, Nebraska, 1961). See also Hippocrates George Apostle, *Aristotle's Philosophy of Mathematics* (Chicago, 1952), p. 12.

[20]*Physics*, IV, 4, 212a 20.

[21]Thomas Greenwood,"The Characters of the Aristotelian Logic," *Thomist*, IV (1942), 240.

[22]*Metaphysics*, XI, 12, 1069a 12; XIII, 8, 1084b 26.

[23]Vincent Smith, *St. Thomas on the Object of Geometry* (Milwaukee, 1954), p. 37.

[24]*Metaphysics*, XIV, 3, 1090b 5-6.

[25]*Metaphysics*, V, 13, 1020a 13.

[26]*Metaphysics*, XIV, 3, 1090b 6-8.

[27]Mansion, since he has wrongly identified mathematical quantities with common sensibles (See footnotes 2 and 122 of this chapter.), says such objects are no more immobile than physical things. (*Introduction à la Physique . . .*, p. 184 and "La physique aristotélicienne . . .," 14-15) He says they are immobile only inasmuch as they are universals abstracted from individuals. But then he points out correctly that even the objects of physics are immobile by abstraction of the essence from the particular mobile physical things, and thus that there is no difference in this respect between the two sciences. However, the point is that even the *particular individual* mathematical objects are immobile according to Aristotle. This is the difference, and this is why immobility is cited by Aristotle as a fundamental difference between physical and mathematical entities. Mansion might have seen this if he had not identified mathematical quantities with common sensibles. The Stagirite's stress on immobility

is not, therefore, simply a carry-over from the Academy as Mansion says. See Mère St. Édouard's critique of this, "La division aristotélicienne des sciences, selon le professeur A. Mansion," Laval Théologique et Philosophique, XV (1959), 225-28.

[28]Ross in Aristotle, p. 62, holds mathematicals to be unchangeable as do Greenwood, Études sur . . ., p. 35 and Diego Pro, "Filosofía de la matemática en Aristóteles," Sapientia, XI (1956), 85. Apostle, in his . . . Philosophy of Mathematics, pp. 19-20, holds the same and logically goes so far as to say (p. 66) that he doesn't want to speak of the addition and subtraction of mathematicals for such words imply change. This seems rather ridiculous, though it is consistent. Father Owens, too, in his The Doctrine of Being . . ., pp. 129 and 209-210, refers to mathematicals as unchangeable. However, in his more recent work, The History of Ancient Western Philosophy (New York, 1959), he calls them immobile, pp. 321-322. Others say, and they are more accurate in my opinion, that mathematicals are immobile: Eduard Zeller, Aristotle and the Earlier Peripatetics, Vol. IV, Philosophy of the Greeks translated by B. F. C. Costelloe and J.H. Muirhead (2 vols.; New York, 1897), 183; M.D. Philippe, "Abstraction, addition, séparation dans la philosophie Aristote," Revue Thomiste, XLVIII (1948), 462-63. Thomas Heath, Mathematics in Aristotle (Oxford, 1949); M.-V. Leroy, "Le savoir spéculatif," and "Annexe: Abstractio et separatio d'après un texte controverse de saint Thomas," Revue Thomiste, XLVIII (1948), and Mansion, Introduction à la Physique . . . and "La physique aristotélicienne . . ." refer to them only as immobile.

[29]Texts in which he calls mathematicals immobile are the following: Physics, II, 2, 193b 33-34, II, 7, 198a 16-18; Metaphysics, I, 8, 989b 32-33, VI, 1, 1026a 8-10, 14, XII, 1, 1069a 34-37, V, 14, 1020b 16-17, III, 2, 997b 19-20; On the Motion of Animals, 1, 698a 25-27, in The Works of Aristotle translated under the editorship of J.A. Smith and W.D. Ross, Vol. V (Oxford, 1949). There are two and possibly three places that I know of in the Metaphysics where the Ross translation refers to the objects of mathematics as unchangeable, viz. III, 1, 996a 21-32, IX, 10, 1052a 5ff., and possibly VII, 10, 1036a 1-12. However, in his more recent translation, Richard Hope, Aristotle's Metaphysics (New York, 1952), shows that the actual Greek terms used by Aristotle in these passages are the same as those which in most other places Ross translates as immobile, viz. some form of the Greek word, kinesis.

[30]Metaphysics, XIII, 2, 1077a 24; also in Metaphysics, XIII, 8, 1084b 9 ff., he speaks explicitly of the generation of angles and numbers.

[31]On the Parts of Animals, translated by William Ogle in The Basic Works of Aristotle edited with an introduc-

tion by Richard McKeon (New York, 1941), II, 1, 646a 31.

"[G]eneration is a process from a something to a something."

That is why form, though it is and is not, is not strictly speaking generated or destroyed; the composite is, Metaphysics, VIII, 4, 1044b 21 ff. Nonliving things can be said to be generated and corrupted as well as living things. Indeed, the whole second book of his On Generation and Corruption is on the generation and corruption of nonliving things.

[32]Metaphysics, I, 9, 991b 21.

[33]Metaphysics, XIII, 8, 1084b 22.

[34]Metaphysics, V, 14, 1020b 7-8.

[35]In both the Physics, III, 6, 207b 1-15 and the Metaphysics, XIII, 8, 1084b 2-23, he speaks of both addition and division of numbers. On this point, see also Apostle, . . . Philosophy of Mathematics, pp. 92-93.

[36]Throughout the Metaphysics, VII, 9, 1034b 20 - 1035b 14, Aristotle speaks of figures divided into other figures. He does this also in IX, 6, 1048a 31-33. In Metaphysics, XIII, 8, 1084b 6-19, he speaks of both generation and division of figures. III, 4, 1001b 13-14 speaks of addition of magnitudes.

[37]Even those authors who say that the Stagirite holds that mathematicals are unchangeable, at the same time admit that he speaks of the operations (addition, subtraction, etc.) by which they are generated and corrupted (though they don't use these words). For example, Pro, "Filosofía de la matemática . . .," says they are unchangeable, 85, and then, 93, speaks of numbers resulting from addition and in general of mathematical quantities arising by combination of their material elements (92-93). Greenwood, Études sur . . ., does basically the same thing. On page 35, he calls mathematicals unchangeable; yet on subsequent pages, 38, 39 ff., he speaks of their coming to be by division and addition of other mathematicals. Apostle, . . . Philosophy of Mathematics, falls into the same contradiction. On pp. 19-20, he calls them unchangeable and yet later (p. 65 ff.), he speaks of their coming from addition and subtraction — though, as I pointed out in footnote 28, he is aware of this contradiction for he says he does not like to even use such terms which imply change.

In general, I think this contradiction is unavoidable for one who holds that the objects of mathematics are unchangeable. Of course, to hold, as I think Aristotle does, that they are immobile only in the strict sense eliminates any difficulty in this matter.

[38]Metaphysics, VIII, 5, 1044b 21-29. Actually, Aristotle points out that there are very definite limitations on the ways in which it is possible for mathematicals to be generated and destroyed. Points which are nondimensional cannot combine to generate a line (which is unidimensional) (Physics, V, 6, 231a 25-26); nor lines (which have no width) placed side by side to form a plane having length and width; nor planes (which have no depth) placed on top of each other to form a solid having length, width, and depth. Likewise two lines cannot be generated from a given line by dividing it in width, for a mathematical line has no width. On the other hand, a line's length can be bisected resulting in the generation of two new lines. Two plane surfaces cannot be generated by dividing a given plane in depth for a mathematical plane has no depth. However, a plane or plane figure could be divided in length and width producing other planes or plane figures. In general, one geometrical figure can generate another only by being divided in those dimensions which it has: a line in length, a plane or plane figure in length and width, a solid in length, width, and depth (a number in nondimensional units only). De Caelo, I, 1, 268a 7-12; Physics, III, 7, 207b 6-10; Metaphysics, III, 4, 1001b 16-17.

Some have interpreted the following passage from Aristotle as saying unqualifiedly that no mathematical objects are generated. (For example, Owens, The Doctrine of Being . . ., p. 138 and Apostle, . . . Philosophy of Mathematics, pp. 19-20) This is the passage (Metaphysics, III, 5, 1002a 33 -b5):

". . .[P]oints and lines and surfaces cannot be in process either of becoming or of perishing, when they at one time exist and at another do not. For when bodies come into contact or are divided, their boundaries simultaneously become one in the one case — when they touch, and two in the other — when they are divided; so that when they have been put together one boundary does not exist but has perished, and when they have been divided the boundaries exist which before did not exist (for it cannot be said that the point, which is indivisible, was divided into two). And if the boundaries come into being and cease to be, from what do they come into being?"

However, I think that all this text is saying, all it could be saying, is that there are certain ways in which lines and surfaces cannot be generated or destroyed, ways which I indicated above. Thus, two plane surfaces which are boundaries of two bodies cannot be generated by dividing in depth the plane surface of another body. The points which are boundaries of lines cannot be generated by dividing another point in two. Aristotle himself substantiates this interpretation when he says in an earlier passage (Metaphysics, III, 4, 1001b 12-17):

"[T]he objects of mathematics, e.g. a plane or a line, added in one way will increase what they are added to, but in another way will not do so . . . . [T]he indivisible [viz. the unit] when added will make the number, though not the size, greater . . . ."

[39]On the parts of a quantum as only potentially present in the quantum until after its division, thus making the whole actually one, cf. Apostle, . . . Philosophy of Mathematics, pp. 55-56; Pro, "Filosofía de la matemática. . .," 92-94.

[40]See the whole of Chapter 10, Book VII of his Metaphysics throughout which Aristotle refers to the parts of mathematicals as their matter. Especially note the following,

". . .[T]he formula of the circle does not include that of the segments . . . the segments are parts in the sense of matter on which the form supervenes. . . ."
(1035a 9-13)

". . .[T]he line when divided passes away into its halves . . . it does not follow that [it is] composed of these as parts of [its] essence, but rather as matter. . . ."
(1035a 18-19)

He does the same in Metaphysics XIII, 8, 1084b 5-14:

". . .[E]ach of the units is part of the number as its matter, and the number acts as form. And in a sense the right angle is prior to the acute, because it is determinate and in virtue of its definition; but in a sense the acute is prior, because it is a part and the right angle is divided into acute angles. As matter, then, the acute angle and the element and the unit are prior, but in respect of the form and of the substance as expressed in the definition, the right angle, and the whole consisting of the matter and the form are prior; for the concrete thing is nearer to the form and to what is expressed in the definition, though in generation it is later."

[41]The references cited in the previous footnote show that, in calling these parts matter, Aristotle means that out of which the composite is formed and into which it can be corrupted. The term matter is used analogously by Aristotle and signifies any substratum from which a thing comes to be and which persists in the result (Physics, II, 7). For example, the bricks, boards, etc. which go to make up a house are called its matter (Metaphysics, VIII, 2, 1043a 14-16). The elements and bodies which go to make up a physical thing and into which it is corrupted are called its matter (On Generation and Corruption, II, 5, 7, 8; for instance, 332a 5-6, 335b 30 ff.) The parts of an animal are called its matter

(On the Parts of Animals, I, 1, 640b 17-29; On the Generation of Animals, translated by Arthur Platt in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), I, 1, 715a 7-12.) On this point, cf. Ross, Aristotle, pp. 73-74. On the analogous character of form and matter for Aristotle, cf. Francis J. Collingwood, Philosophy of Nature (Englewood Cliffs, New Jersey, 1961), Chapter 6 — "Matter and Form are Correlatives," "Aristotle's Hylemorphism," and "Formal and Material Parts"; see also Apostle, . . . Philosophy of Mathematics, p. 50 ff.

[42]Metaphysics, VII, 10, 1036a 1-13. It is very clear in the Metaphysics that the intelligible matter of the objects of mathematics is identified with the parts of these objects (i.e. the segments of lines and figures, the units of number). In fact, it is in the very same chapter (Chapter 10 of Book VII) where he calls the parts of mathematicals their matter that he calls this matter "intelligible." If there is any doubt that Aristotle identifies intelligible matter with the material parts of mathematical objects a reading of this chapter will remove it.

A paragraph in the following chapter, Chapter 11, also very clearly calls the parts of a circle, its semicircles, its matter and calls this matter intelligible (1036b 32 - 1037a 4). Father Owens in his The Doctrine of Being . . ., pp. 209-210, also points out the identification by Aristotle of the parts of mathematicals with intelligible matter as do Gabriel, "Matière intelligible . . .," 191; Pro, "Filosofía de la matemática . . .," 92-94; and Apostle, . . . Philosophy of Mathematics, pp. 151-152.

[43]As Yves R. Simon, "The Nature and Process of Mathematical Abstraction," Thomist, XXIX (April, 1965), points out, 131, the term intelligible matter is also used by Aristotle in reference to an element of a definition. He speaks of the genus being the matter of the species in relation to the difference as form.

> "Of matter some is intelligible, some sensible, and in definition there is always one element matter, and the other act, e.g. the circle is a 'plane figure.'"
> (Metaphysics, VIII, 6, 1045a 33-36)

In fact, it is because the parts of a definition are related as matter to form (potency to act) that the definition is a unity. (Metaphysics, VIII, 6) The extension of this notion of intelligible matter to the logical order seems to me to be simply an analogous use of the term. Both the "matter" of a mathematical, and the "matter" of a definition function as potency, as determinable, to some further formal specification. Also both matters are in the mental (or intelligible) rather than the real order. (To see that the intelligible matter of mathematicals is only in the mental order, see my

text, pp. 73-75. Cf. also footnote 154 below. See Owens' discussion of this in The Doctrine of Being . . ., pp. 209-210 and W.D. Ross, Aristotle's Metaphysics, II (Oxford, 1958), pp. 199-200.)

[44]Of course, for the substantial changes of generation and corruption, there must ultimately be some kind of prime matter present as the most basic substrate. A determination of the most basic matter underlining the "substantial" change of mathematicals will be deferred until we talk directly about the nature of their "intelligible" matter. (See below, footnote 154.)

[45]If this is all that is meant by those authors who say that mathematicals for Aristotle are unchangeable, then I have no quarrel with them.

[46]See, for example, Categories, 14, where he lists generation and destruction as two kinds of movement. (15a 14)

[47]Metaphysics, VIII, 3, 1043b 17-19. Other places where he says that the form is per se ungenerated are VIII, 1, 1042a 30, VIII, 4, 1044b 22, XII, 1, 1069b 25, and XII, 2, 1070a 15.

[48]Metaphysics, VII, 8, 1033b 17-19.

[49]Physics, for example, though it studies composites of form and matter studies form primarily, since the form is the actuality of the composite. Aristotle emphasizes this in many places. (Cf. for example, On the Parts of Animals, I, 1, 640b 18-30, I, 1, 641a 18-33; Metaphysics, VII, 11, 1037a 13-18.) Recall that nature is primarily form, Physics, II, 1, 193a 35 - b13. On this point, see Owens, "The Aristotelian Conception of the Sciences," International Philosophical Quarterly, IV (May, 1964), 213-214 and "Our Knowledge of Nature," 71-72. On the necessity of including form in a complete study of natural things, cf. Collingwood, Philosophy of Nature, Chapter 6, section entitled "Formal Causes Cannot Be Omitted in a Full Account of Physical Being."

[50]If it is objected that in this sense even physics could be said to be primarily of that which is not per se generated and corrupted, for it primarily studies the forms of natural things, I would reply that this is true. However, inasmuch as the physicist studies these forms with reference to matter and motion (Metaphysics, XI, 1, 1059b 16 ff.; XI, 7, 1064a 15-27; XI, 3, 1061b 7, 28-32), physics taken in its totality and not just in reference to what it primarily studies is not of the ungenerable and the incorruptible. The same could be said of mathematics. (Incidentally, even Thomas Aquinas makes this point in reference to physics — cf. De Trinitate, V, 2c [p. 21 Maurer trans.])

[51]Philippe, "Abstraction . . .," 464. Pro, "Filosofía de la matemática . . .," 98, also apparently holds this as does Ross who says that mathematical essences are pure form, Aristotle, pp. 158, 167-8. This also is apparently the position Father Owens takes in his The Doctrine of Being . . ., pp. 209-210.

[52]Posterior Analytics, I, 13, 79a 8.

[53]There are passages where Aristotle refers to the objects of mathematics as separate from matter (e.g. Metaphysics, II, 3, 995a 15-16; VI, 1, 1026a 6-15). However, these alone do not necessarily mean that mathematical objects are pure forms. The matter from which they are said to be separate may only refer to sensible matter; they may still be forms plus intelligible matter (Individual mathematicals certainly do have this matter, as we have seen.). Granted that mathematical essences are separate from sensible matter, i.e. matter subject to sensible qualities, our question is whether or not they are pure forms separate from all matter, intelligible matter as well as sensible.

[54]Metaphysics, XI, 1, 1059b 25.

[55]It is the essential nature expressed in the definition, Posterior Analytics, II, 3, 91a 1, not the individual, which is the foundation of science (Posterior Analytics, II, 3, 90b 24, II, 17, 99a 22-23). On this point, see Owens, "The Aristotelian Conception . . .," 201-204.

[56]Metaphysics, VII, 10, 1035a 9-13.

[57]On the identification of the term formula with the term definition, see Owens, The Doctrine of Being . . ., p. 391, n. 18. In many places, Heath in his translations in Mathematics in Aristotle uses the term definition where the Ross translation uses the word formula. Compare their translations of Metaphysics, VII, 10, 1035b 33-36 (Heath, p.214) and VII, 11, 1036b 32-1037a 5 (Heath, p. 215).

[58]Posterior Analytics, II, 3, 91a 1.

[59]Metaphysics, VII, 10, 1035a 30- b2.

[60]Metaphysics, VII, 11, 1037a 1-2. Heath has "[T]here is some sort of matter in everything which is not an essence and a pure form but is some individual thing." [underlining mine] Mathematics in Aristotle, p. 215.

[61]Metaphysics, VII, 10, 1035b 9-14.

[62]Metaphysics, VII, 11, 1037a 3-4.

[63]A text of the De Caelo, I, 9, 278a 3 - 278a 10

also apparently relegates matter only to the particular individual circle. Aristotle says,

"[T]he being of circle and the being of this particular circle are different things, the one being form, the other being the form (embodied) in the matter, i.e. a particular thing." (Heath translation, Mathematics in Aristotle, p. 213)

It is worth noting that W. D. Ross in his commentary on Chapter 10 of Book VII of the Metaphysics also interprets Aristotle as denying matter of the essence of circle, Aristotle's Metaphysics, I, pp. 197 and 203.

[64]It is interesting to note Thomas Aquinas's commentary on these passages of Aristotle having to do with the matter of a circle.

1. Concerning the first one (pp. 38-39 of my text), Thomas interprets Aristotle as saying that the matter proper to individual circles (in other words, individual matter) is not included in the expression of the form of circle. (Commentary on the Metaphysics of Aristotle, translated by John P. Rowan in "Library of Living Catholic Thought," I [Chicago, 1961], VII, L. 9, 1475) (And he adds later in that lesson that forms of this type must include common matter by addition in their definition. 1477, 1478) However, if Aristotle's expression "formula of the circle" can be taken to mean "definition of the circle" (And in many places, formula and definition are used synonomously by him, cf. note 57.), then he is saying much more than Thomas acknowledges. He is saying the definition which expresses the essence does not include matter.

2. Concerning the second (p. 39 of my text), Thomas interprets Aristotle as referring again to individual matter of this circle and denying that it is included in the species of circle (In VII Metaphysics, L. 9, 1480-1); he interprets "circle unqualified" to mean species of circle. (And he has stated earlier (In VII Metaphysics, L. 9, 1477-8) that common matter must be included therein.) But in his passage, the Stagirite has not only not mentioned common matter, he has said simply that matter is part of the concrete thing as if this is the only matter there is. Furthermore, "circle unqualified" seems to be clearly identified with form of circle alone, and this would seem to be identical with the essence of circle for it is the only meaning of circle given other than the particular circle.

3. Thomas' commentary on the third (pp. 40-41 of my text) is the most interesting. Once again he interprets Aristotle as referring to individual matter of this circle and denying it of the species circle (In VII Metaphysics, L. 12, 1572). But where Aristotle's text reads, "there is some matter in everything which is not an essence and a bare form," clearly identifying essence and form, Thomas' text reads, "[T]here is some matter in everything which is not an essence and a species . . . ." (In VII Metaphysics, L. 12, #636 of Aristotle's text) It is quite logical then for him to reason that Aristotle is simply denying individual matter of the species and essence (the universal circle). However, as I discussed in my text, p. 41, Aristotle seems rather to be unqualifiedly denying matter, and making no distinction between individual and common matter, of the essence which he identifies with "bare form" and "universal circle."

In his commentary then, aside from the fact that in one place he has a bad translation, Thomas introduces a distinction which I simply cannot find here in Aristotle, viz. a distinction between individual and common matter. And so he consistently interprets the Stagirite as denying individual matter of the essence, the species, the definition, of circle. However, Aristotle in every text speaks just of matter without qualifying it. And matter unqualified is denied of the formula (definition?) of circle, (text #1), of "circle unqualified," which seems to be essence (#2), and of the universal circle, essence and form which he identifies (#3).

[65]Metaphysics, VII, 10, 1035a 17-20.

[66]The term "straight" in a mathematical context is sometimes used by the Greeks rather than our term straight line. They would probably consider our addition of the word line redundant. Other places where Aristotle uses the term "straight" to mean straight line are De Anima, I, 1, 403a 12-16; Metaphysics, III, 2, 997b 35 - 998a 4.

[67]De Anima, III, 4, 429b 13-14.

"[F]or flesh necessarily involves matter and is like what is snub-nosed, a this in a this."

[68]De Anima, III, 4, 429b 17-20.

Thomas Aquinas' commentary on these passages follows much the same pattern as his commentary on the ones having to do with the circle. (Footnote 64)

1. On the first, (page 42 of my text) he as usual interprets Aristotle as saying that individual

matter is not part of the definition of line, In VIII Metaphysics, L. 9, 1476, but only of the individual line. And immediately following, he adds that definitions of both forms and composites must include common matter. Aristotle's text on the other hand does not distinguish these two kinds of matter but simply denies matter of the essence line.

2. On the text of the De Anima (p. 42 of my text) (Aristotle's "De Anima" in the Version of William of Moerbeke and the Commentary of St. Thomas Aquinas, translated by Kenelm Foster, O.P., and Silvester Humphries, O.P. with an introduction by Ivo Thomas, O.P. (London, 1931), III, L. 8, 714), he again interprets Aristotle as denying only individual matter of the essence of straight line. (Actually the translation he has does not include the term matter in it.) Aristotle, however, uses the term matter (not individual matter) and affirms it only of the individual line.

[69]Metaphysics, VII, 12, 1035b 6-14; 20-23.

In his commentary on this text (In VII Metaphysics, L. 10, 1483), Thomas interprets Aristotle's expression "formula of the right angle" to mean intelligible expression of the form. And he then proceeds to say that the Stagirite is simply denying matter of this expression. However, formula can mean definition for Aristotle (See note 57.) and so the Greek philosopher may be saying much more, viz. that the definition of right angle (and hence the essence expressed thereby) does not include matter.

[70]Smith, The General Science . . ., p. 18, claims that Aristotle holds that mathematical definitions (and hence essences) include common intelligible matter. However, Aristotle never says this. Interestingly enough the text Smith cites in support of this statement is from Thomas Aquinas' Commentary on the De Trinitate (p. 25, n. 45)! Mère St. Édouard also says the same, "La division . . .," 219, 222, 232 as do Gabriel, "Matière intelligible . . .,"1, 185-186 and Apostle, . . . Philosophy of Mathematics, p. 53; but none of them give any text which really proves this. (There are none!) Heath, on the other hand, in his Mathematics in Aristotle, pp. 214-216, seems to hold that matter is proper only to the individual mathematicals, not to the universal expressed in the definition. Needless to say, it is very important, indeed it is essential if one is to understand Aristotle's own doctrine, to read Aristotle as Aristotle (or at least attempt to), not through the eyes of men centuries later.

[71]Posterior Analytics, II, 11, 94a 28-35. Also see Chapter IV, pp. 125-6.

[72]*In II Posterior Analytics*, L. 9, 5, translated by Pierre Conway, O.P. (Quebec, Canada, 1956). For Thomas's use of this proof and comments on it see Part II, Chapter IV, pp. 271-3.

[73]The commentators are certainly not in agreement on this point. Ross, *Aristotle*, p. 69 ff., holds that Aristotle's position is that sensible essences do contain common sensible matter. as do Mère St. Édouard, "La division . . .," 219-221, Leroy, "Le savoir spéculatif . . .," 329, and Edward Maziarz, *The Philosophy of Mathematics* (New York, 1950), p. 40. Father Armand Maurer, "Form and Essence in St. Thomas," *Medieval Studies*, XIII (1951), 167-169, and Father Owens, *The Doctrine of Being . . .* hold that they do not. Much of what follows in my paper on this point has been gained from Father Owens' excellent analysis. I refer the reader to his Chapter 10 of *The Doctrine of Being . . .* especially.

[74]*Metaphysics*, VII, 7, 1032b 1. I prefer to follow Father Owens in translating the Greek *ousia* as entity. My reasons are the same as his (*The Doctrine of Being . . .*, Chapter 4). The various English translations will render this Greek term often as substance, frequently as essence, occasionally as form, depending on the context. Such diversity in translating the same Greek word makes an accurate textual analysis practically impossible. Therefore, I will stay with the *one* term, entity.

[75]*Metaphysics*, VII, 10, 1035b 33-34.

[76]At *Metaphysics*, VIII, 2, 1043a 27-28, Aristotle speaks of three kinds of sensible entity,

> "[O]ne kind of it as matter, another as form or actuality, while the third kind is that which is composed of these two."

Of these three, the "entity without matter," referred to in the passage in my paper, p. 45, could only mean the form. Also see footnote 92 for the notion of form as entity.

[77]*Metaphysics*, VII, 7, 1032b 14.

[78]*Physics*, II, 2, 194b 10-11.

[79]*Posterior Analytics*, II, 3, 91a 1: *Metaphysics*, VII, 5, 1031a 12.

[80]*Metaphysics*, VII, 11, 1036a 29.

[81]*Metaphysics*, VII, 11, 1036a 27 - b5.

[82]In this very passage, he seems to use the two terms, definition and formula, interchangeably. Also see above

footnote 57.

[83]Metaphysics, VII, 12, 1038a 26.

[84]Metaphysics, VII, 12, 1038a 19-20.

[85]Both Father Owens, The Doctrine of Being. . ., pp. 223-224 and Father Maurer, "Form and Essence . . .," 166-169 agree that this is Aristotle's position. It is interesting to note also that the classical Greek commentators on Aristotle (Alexander of Aphrodisias, Asclepius, and Themistius) all say that for the Stagirite essence is form as distinct from matter. (Cited by Maurer, op. cit., 168, note 16.) Averroes, of course, says the same. (Cf. Maurer, op. cit., section I.)

The fact that in every definition according to Aristotle the genus is as matter to the specific difference (See above, footnote 49.) does not in my opinion rule out the possibility that a definition may express only the form. (Indeed, as we have just seen, Aristotle does state that some definitions are only of form.) The Stagirite points out that the form has parts and that these parts must be included in its definition. (Metaphysics, VII, 10, 1035b 3ff. — "[T]he parts of the form are parts of the definition . . . ," Heath's translation, Mathematics in Aristotle, p. 214) Some parts of the form could be expressed in the definition as "matter" in relation to other parts as form. Hence, the definition would include intelligible matter (meaning genus in relation to difference) even though it would be only of form.

[86]Physics, II, 2, 193b 35 - 194a 6.

[87]Metaphysics, VI, 1, 1025b 26 - 1026a 5.

[88]Metaphysics, VII, 5, 1031a 9-10.

[89]Metaphysics, VII, 4, 1030a 29-31.

[90]Metaphysics, VII, 4, 1030b 5-6.

"[D]efinition and essence in the primary and simple sense belong to entities. Still they belong to other things as well, only not in the primary sense."

[91]Metaphysics, VII, 7, 1032b 1.

[92]Metaphysics, VII, 11, 1037a 28-29.

"[T]he entity is the indwelling form, from which and the matter the so-called concrete entity is derived."

In many places form, composite, and even matter are called

entity. (Metaphysics, VIII, 2, 1043a 27-28, text given in footnote 76) However, form is obviously primary for it accounts for the matter being part of the composite and for the composite being the definite actual thing it is. (Metaphysics, VII, 3, 1029a 27-33; VII, 3, 1029a 5-7; VII, 17, 1041b 7-8, 25-27.) On this point, see Owens, The Doctrine of Being . . ., pp. 200-205.

[93]Actually in the De Anima, I, 1, 403b 7-16 and Metaphysics, VIII, 2, 1043a 12-21, Aristotle indicates that there are three possible definitions of a thing: definition of form, of form and matter, of matter.

"Obviously, then, the actuality or the formula is different when the matter is different; for in some cases it is the composition, in others the mixing, and in others some other of the attributes we have named. And so, of the people who go in for defining, those who define a house as stones, bricks, and timbers are speaking of the potential house, for these are the matter; but those who propose 'a receptacle to shelter chattels and living beings,' or something of the sort, speak of the actuality. Those who combine both of these speak of the third kind of substance, which is composed of matter and form (for the formula that gives the differentiae seems to be an account of the form or actuality, while that which gives the components is rather an account of the matter) . . . ."

That there are three definitions is to be expected since all three — form, matter, and individual composite, are called entity. (Metaphysics, VII, 2, 1043a 27-28) Ross makes the same point, Aristotle's Metaphysics, I, p. civ.

[94]Metaphysics, VII, 10, 1035b 33 (Heath's translation, p. 214, Mathematics in Aristotle).

"Only the parts of the form are parts of the definition . . . ."

And at VIII, 1, 1042a 28-30, he says that form "being a 'this' can be separately formulated." (For the meaning of "this" see Chapter I, pp. 8-9.) Some of the other places where the Stagirite speaks of the definition of form alone are Metaphysics, III, 2, 996b 6-8; VII, 11, 1037a 22 ff.; VIII, 2, 1043a 12-21; VII, 10, 1035a 20-21. On this point, see Owens' analysis of Aristotle's statement that the form is "separate in notion", The Doctrine of Being . . ., pp. 238-241. Owens shows that "separate in notion" means "separate in intelligible content." Ross also notes in his book on the Metaphysics that Aristotle speaks of defining form alone — Aristotle's Metaphysics, I, pp. clii-cvi and II, p. 199.

St. Thomas Aquinas holds, on the contrary, that no

substantial form of a physical thing can be defined without including matter at least by addition in the definition. (In II De Anima, L. 1, 213; In VII Metaphysics, L. 9, 1477) St. Thomas notes, that, as a matter of fact, Aristotle too, when he defines the substantial form which is the soul, does include matter in its definition. Be that as it may, the Stagirite does assert, as the above cited texts show, that form can be defined without matter — whether or not he ever so defines it is another question — but he says it can be done. In other words, it seems to me that we must distinguish between what Aristotle says can be done and what as a matter of fact he does. Indeed, if Thomas is right on this point, the Stagirite can never do what he says.

[95]See the texts cited in footnote 92.

[96]Metaphysics, VII, 11, 1037a 5-7.

"It is clear also that the soul is the primary entity and the body is matter, and man or animal is the compound of both taken universally . . . ."

See the text cited by footnote 98 also.

[97]In VII Metaphysics, L. 12, 1523-24.

[98]Metaphysics, VII, 10, 1035b 28-32. On this point see Owens, The Doctrine of Being . . ., pp. 203-205.

[99]See the texts of note 93.

[100]Philippe, "Abstraction, . . .," cites two such places: De Caelo, III, 1, 299a 16; Metaphysics, XIII, 2, 1077b 10.

[101]See Philippe's excellent discussion of definition by addition, "Abstraction . . .," 466-468. Maurer, "Form and Essence . . .," 168-169 and Owens, The Doctrine of Being . . ., pp. 238-240 also discuss this.

[102]Maurer, "Form and Essence . . .," 167-169 so interprets Aristotle's definition by addition. See also footnote 85.

[103]We discuss the definition of accidents on pp. 67-68. Cf. Philippe, "Abstraction . . .," 467 and Ross, Aristotle's Metaphysics, I, p. xcviii.

[104]Philippe, "Abstraction . . .," so interprets Aristotle's definition by addition, 468 and 477.

[105]To definitively resolve this problem would take at least an extensive textual analysis to determine precisely Aristotle's meaning of definition, formula, and also what exactly the notion "includes matter" means when said of def-

initions of physical things. Since I lack a knowledge of Greek, I cannot undertake such an analysis, though as I have said (footnote 73), even the authorities are not in agreement on this point. Actually, it may well be that Aristotle himself is undecided. (Ross and Maurer incline toward this opinion. Cf. Maurer, "Form and Essence . . .," 175-176 and Ross, Aristotle's Metaphysics, I, pp. ciii-cvi.)

[106]Metaphysics, VII, 10, 1035a 17-23.

[107]Metaphysics, VII, 10, 1035a 30 - 1036a 12.

[108]Metaphysics, VII, 7, 1032a 15-22; XII, 2.

[109]Metaphysics, I, 8, 989b 29-31.

[110]Metaphysics, XI, 1, 1059b 11-13. See Philippe, "Abstraction . . .," 469-474, for the various meanings of the term separate or separable for Aristotle.

[111]Metaphysics, XI, 7, 1064a 31-33.

[112]Metaphysics, XIV, 3, 1090a 28-29.

[113]Physics, II, 2, 193b 23-24.

[114]Metaphysics, XI, 2, 1060b 12-17.

[115]Metaphysics, XII, 1, 1069a 25.

[116]Categories, 5, 2b 4-6; Pro, "Filosofía de la matemática . . .," 81, 89.

[117]Metaphysics, XIII, 3, 1077b 11-17.

[118]See the whole of Chapter 19, Book II of the Posterior Analytics.

[119]Posterior Analytics, I, 18, 81a 40 -b9.

"Thus demonstration develops from universals, induction from particulars; but since it is possible to familiarize the pupil with even the so-called mathematical abstractions only through induction — i.e. only because each subject genus possesses, in virtue of a determinate mathematical character, certain properties which can be treated as separate even though they do not exist in isolation — it is consequently impossible to come to grasp universals except through induction. But induction is impossible for those who have not sense-perception. For it is sense-perception alone which is adequate for grasping the particulars: . . ."

[120]De Anima, III, 1, 425a 14-18.

[121]Nicomachean Ethics, translated by W.D. Ross in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), VI, 8, 1142a 26-29.

"[P]ractical wisdom is concerned with the ultimate particular, which is the object not of scientific knowledge but of perception — not the perception of qualities peculiar to one sense but a perception akin to that by which we perceive that the particular figure before us is a triangle;"

I agree with Heath's interpretation of this passage; he says, Mathematics in Aristotle, pp. 276-277, that it states that mathematical objects, like triangles, are perceived as common sensibles.

[122]W. Kane stresses this point in "Abstraction and the distinction of the sciences," Thomist, XVII (1954), 60 as do Pro, "Filosofía de la matemática . . .," 96-98 and C. De Koninck, "Abstraction from Matter, II," Laval Théologique et Philosophique, XVI (1960), 58. On the other hand, Mansion holds that mathematics does really deal with common sensibles, "La physique aristotélicienne . . .," 14-15 and Introduction à la Physique . . ., p. 167 ff. The analysis in my text, pp. 59-60, shows him to be wrong. See also Mère St. Édouard's criticism, "La division . . .," 223-228.

[123]Physics, II, 2, 193b 23-24; 32-34.

[124]Owens, The Doctrine of Being . . ., p. 239, p. 397, n. 30.

[125]A.E. Taylor, Aristotle on His Predecessors, translated from the text edition of W. Christ with introduction and notes (2d ed.; Chicago, 1910), p. 25; Philippe, "Abstraction . . .," 464; Leroy, "Le savoir speculatif," 330-331.

[126]De Anima, III, 7, 431b 16-17.

[127]Metaphysics, XIII, 3, 1077b 20 - 1078a 5.

[128]Metaphysics, XIII, 3, 1078a 23-29.

"For a man qua man is one indivisible thing; and the arithmetician supposed one indivisible thing, and then considered whether any attribute belongs to a man qua indivisible. But the geometer treats him neither qua man nor qua indivisible, but as a solid."

Notice in this passage how the arithmetical unit seems to be identified with the substantial unit. (See above, Chapter I, page 6.)

[129]I call this act of abstraction a mental act here simply because it is the most general term I can think of.

It remains to be seen exactly by what power this act takes place. This will be discussed in Chapter III.

[130]Philippe, "Abstraction . . .," points out that the technical term abstraction which Aristotle uses for the mental process of obtaining the objects of mathematics signifies etymologically "subtraction," taking away, removing. It is said with reference to both physical actions (e.g. the sculpturing of a statue involves the abstraction of stone) and mental acts, 461-62. On the notion of abstraction, see also Owens, The Doctrine of Being . . ., pp. 239-40.

[131]Metaphysics, XI, 3, 1061a 29-36.

[132]Nicomachean Ethics, VI, 8, 1142a 17.

[133]Many others have said the same thing. To cite a few: Owens, The Doctrine of Being . . ., p. 267:

"The mathematicals, consequently, have no actual Being outside the thought of the mathematician . . . ."

See also Philippe, "Abstraction . . .," 462; Greenwood, Études sur . . ., pp. 36-37; Ross, Aristotle's Metaphysics, II, 418; Leroy, "Le savoir spéculatif," 329-330; Pro, "Filosofía de la matemática . . .," 85; Giuseppe Alvarez, "La filosofia della matematica in San Tommaso," Revista di Filosofia neo-Scholastica, XLII (1950), 151.

[134]Metaphysics, XIII, 3, 1078a 29-31. For a discussion of this, see Pro, "Filosofía de la matemática . . .," 89-91 and Ross, Aristotle's Metaphysics, II, 418. Alvarez, "La filosofia della matematica . . .," 151 and Owens, The Doctrine of Being . . ., p. 267 also mention this point.

[135]Greenwood, Études sur . . ., p. 36. Pro, "Filosofia de la matematica . . .," uses the same expression, 78 and 85.

[136]To determine the exact type of analogy used in the one case of referring to accidents as secondary entities and in the other of referring to beings of reason (in the sense explained) is outside the scope of this paper.

[137]Metaphysics, VII, 4, 1030a 18 -b13 and all of Chapter 5.

[138]Metaphysics, VII, 5, 1031a 1-3. See also Metaphysics, VII, 13, 1038b 27-28.

"[N]either in formula nor in time nor in coming to be can the modifications [i.e. the accidents] be prior to the substance; for then they will also be separable from it."

See above, pp. 51-53, where we first discussed definition

by addition.

[139]Pro, "Filosofía de la matemática . . .," makes the same point, 89.

[140]Metaphysics, XIV, 5, 1092b 18-20.

[141]De Anima, III, 7,431b 13-17.

[142]Geiger, "Abstraction et separation . . .," 34-35 has an extensive discussion of this point and contrasts Aristotle with Aquinas on this matter. Mansion, "La physique aristotélicienne . . .," also points out Aristotle's lack of doctrine on this point, 15.

[143]Metaphysics, XIII, 3, 1078a 17-19.

[144]Mansion, "La physique aristotélicienne . . .," 15, and Geiger, "Abstraction et separation. . .," 34-35, point out that Aristotle never does establish the priority in reality of quantity preceeding quality.

At the most, there are only very vague indications in his works that Aristotle recognizes the ontological priority of quantity to other accidents. One such indication, whatever its value, is that in his lists of the categories he often lists quantity immediately after substance — thus "substance, quantity, quality . . . ." On the other hand, this should be set against the following statement where he seems to clearly state that quality follows first after substance.

"For if the universe is of the nature of a whole, substance is its first part; and if it coheres merely by virtue of serial succession, on this view also substance is first, and is succeeded by quality, and then by quantity."
(Metaphysics, XII, 1, 1069a 19-22)

Those like Mère St. Édouard, "La division . . .," 219, and Leroy, "Le savoir spéculatif," 331-332, and Gabriel, "Matière intelligible . . .," 190, who find Aristotle giving a reason why abstraction of quantity is possible are, in my opinion, simply reading into his text, reading in St. Thomas' doctrine.

[145]Metaphysics, VII, 5, 1030b 23-24. See the whole of this chapter.

[146]Metaphysics, VII, 3, 1029a 20-22.

"By matter I mean that which in itself is neither a particular thing nor of a certain quantity nor assigned to any other of the categories by which being is determined."

[147]Mansion, "La physique aristotélicienne . . .," 13-14, feels that it is an abuse of language to speak of intelligible matter in reference to the objects of mathematics. Quantity, he says, is no less sensible than the objects of physics. His error, as I have said before (footnote 122) comes in identifying the object of mathematics with quantity, the common sensible. See the criticism of this position by Mère St. Édouard, "La division . . .," 228-233.

[148]Mère St. Édouard, "La division . . .," 226, and Gabriel, "Matière intelligible . . .," 185, 187, and 191, seem to think that Aristotle does say that intelligible matter is identified with substance, subject of quantity and that this is retained by the mathematician after abstraction. Needless to say, they can offer no text to verify this.

[149]The three places, the only places, where Aristotle actually refers to intelligible matter by name are Metaphysics, VII, 10, 1036a 9-10, VII, 11, 1037a 4, and VIII, 6, 1045a 34, 36. Only the first two refer to it in the context of mathematics. The last refers to the genus of a definition (cf. above, footnote 43). On the possible relation between the two meanings, cf. Ross, Aristotle's Metaphysics, II, pp. 199-200.

[150]Metaphysics, VII, 11, 1036b 35 - 1037a 4. See Heath's translation, footnote 60 above.

[151]Metaphysics, III, 6, 1002b 14-16.

[152]Metaphysics, VII, 8, 1034a 5-8.

"And when we have the whole, such and such a form in this flesh and in these bones, this is Callias or Socrates; and they are different in virtue of their matter (for that is different), but the same in form; for their form is indivisible."

Metaphysics, XII, 8, 1074a 33-34. De Caelo, I, 9, 277b 31-278a 11. Cf. Ross, Aristotle's Metaphysics, I, pp. cxv - cxix.

[153]Owens, The Doctrine of Being . . ., p. 209; Ross, Aristotle, p. 69.

[154]N.B., do not confuse intelligible matter, the principle of individuation of mathematicals, with that matter which individuates physical things. Since mathematical quantities exist actually as such only in being known, intelligible matter actually performs its role of individuating only in the mental order. Outside the mind intelligible matter exists only potentially inasmuch as a physical substance is able to be considered purely as a substrate lacking all sensible and other features. (Metaphysics, VII, 10, 1036a 1-13) Inasmuch as this matter functions only in the mental not in the real

order, this gives further meaning to its designation as "intelligible" matter. Exactly what power can be said to grasp individual mathematicals will be discussed in the following chapter.

Incidentally, if I am correct in identifying substance as subject of quantity with intelligible matter as substrate of mathematical forms, then it is this matter which would be the most basic matter underlying the "substantial" changes of mathematicals. See above, footnote 44. On intelligible matter as individuating, cf. Geiger, "Abstraction et separation . . .," 36-38, and De Koninck, "Abstraction from Matter,II," 64-65.

155 Topics. translated by W. A. Pickard-Cambridge in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), in talking about predicates which are of the essence of their subject, Aristotle affirms that such predicates must be of the same category as their subject, I, 9, 103b 38-40.

"[I]f . . . one kind of predicate is asserted of another kind [and kind in the context clearly means category] , it does not signify essence . . . ."

See his whole discussion of this, I, 9.

156 Metaphysics, VII, 6, 1031b 24-27. For the same point, cf. Metaphysics, VII, 4, 1029b 24 - 30a 1. Also see Philippe's discussion of definition by addition, "Abstraction . . .," 467 and Ross, Aristotle's Metaphysics, I, pp. xcviii-ix.

157 Mère St. Édouard, "La division . . .," gives this as Aristotle's teaching, 232. In general, she reads much of what is explicit in Aquinas into the Stagirite. See the points discussed in footnotes 70, 144, and 148.

CHAPTER III

THE ROLE OF IMAGINATION IN MATHEMATICS

We have already mentioned at some length the act of abstraction performed by the mathematician. I want in this chapter to spend some time on what is involved in the psychological process of abstraction, and in particular on what the role of imagination is.

In the famous text of the Posterior Analytics where he describes the general procedure of obtaining the universal from sense experience,[1] Aristotle refers to the presence of what he there calls memory. Animals which have memory, he says, are able to retain sense impressions and so provide for themselves some stability in the changeable data of sense experience. Actually what Aristotle there calls memory he will later more precisely designate imagination.[2] Thus the role of imagination in all abstraction (if I may use this term now in a wider application meaning the mental act of obtaining the universal from the sensible particular) is evident. This would mean, of course, that imagination is present in mathematical abstraction, too, for it also begins with perception of changing sensible particulars. However, and this is a point which should be emphasized, Aristotle never refers to imagination as having a particular or special part in mathematics or mathematical abstraction.[3] (This

should be emphasized because we shall see that Thomas Aquinas does give it a special role in mathematics.)

Now it may be that Aristotle simply did not develop this point, or, on the other hand, it may be that he did not think imagination had any special role in mathematical abstraction. An elaboration of this second possibility is in order. According to the Stagirite, imagination "has the objects of sense for its object."[4] Imagination is said to be the act of a sense faculty (though just what sense faculty is not clear)[5] and is clearly distinguished from the acts of the mind, affirmation and negation and the knowledge of incomposites.[6] But as we have noted, the objects of mathematics according to the Stagirite are not sensible for the mathematician leaves out the proper sensibles. Though his abstraction is based upon perception of the common sensibles, it is not these qua sensible which he studies. (Chapter II, pages 59-63) Since mathematical quantities are not sensible, it would seem to follow that they could not be imagined either, for imagination as we said is the act of a sense power, it has "the objects of sense for its objects." Mathematicals, then, would be knowable only by the mind. We could point out in support of this last statement that Aristotle calls the matter of mathematicals "intelligible;" he never refers to it as "imaginable." However, we should also add that the Stagirite never categorically denies that the objects of mathematics are imaginable.

Indeed, from another point of view, it would seem that the very notion of intelligible matter indicates that

mathematical quantities must be imaginable. Recall that for Aristotle intelligible matter is a principle of individuation of mathematical forms. Individuals, however, are attained directly only by sense, and not by mind which is directly of the universal.[7] Hence individual mathematicals, quantitative forms in individual intelligible matter, would have to be grasped directly by a sense faculty. But the quantities studied in mathematics are not colored, tangible, etc., as we have seen; they lack all proper sensibles and are not actually but only potentially in sensible things. Therefore they cannot be grasped by the exterior senses. Could it be imagination which grasps them?

In a very interesting text of his Metaphysics, the Stagirite speaks of individual mathematicals as known by "intuition."

> But when we come to the composite thing, e.g. this circle, i.e. one of the singular circles, whether sensible or intelligible (I mean by intelligible circles the mathematical, and by sensible circles those of bronze, or of wood) — of these there is no definition, but they are known with the aid of intuition or of sensation; and when they pass out of this actual cognition it is not clear whether they are or not; but they are always expressed and known by the universal formula.[8]

Is this intuition imagination? Some have so interpreted it.[9] This much is clear from the text; it is not an act of direct sensation nor is it an act of mind, that which grasps the universal formula. In the absence of statements to the contrary, the logical alternative is that it is imagination which is meant.[10] Exactly how such entities could be imaginable, in view of the fact that they lack sensible quali-

ties, still remains to be seen.

Indeed, if the imagination is not the "location" of the individual mathematical quantities, where would the mathematician find the appropriate image from which to abstract the universal mathematical essence? Appropriate singulars from which physical natures can be abstracted can be supplied by the external senses; all imagination apparently does, according to Aristotle, is to stabilize the sense impressions. In the case of mathematical natures, however, the individual mathematicals from which the mind could abstract these natures are not found in act in nature; hence they can't be supplied by the external senses. Where then can they be found? There is certainly room here for a sense power such as the imagination to supply the needed individuals. If such is the case then imagination would have much more of a role in mathematics than simply stabilizing sense impressions. Indeed, since the individual mathematicals which it would contain lack sensible qualities and sometimes one or more dimensions, it would have to mean that imagination itself had performed a subtraction (abstraction) upon the sensed objects leaving aside certain of their sensible features. In other words, if the individual mathematical quantities are in the imagination (and I say if for as I have said this is not mentioned explicitly by Aristotle), the abstraction which the mathematician performs would involve at its very root an act of imagination leaving aside sensible features of physical things. The mathematician would thus end up possessing in imagination individual mathematical quantities

from which, using the faculty of mind, he could proceed to abstract mathematical essences.

Now Aristotle does say that imagination in man need not simply be a reproduction of an originally sensed object but is free to manipulate the stored sense impressions, the images. He speaks of "deliberative" imagination as present only in man.[11] (This apparently refers to imagination under the direction of reason, for deliberation, he says, is an action of the reason.)[12] By means of deliberative imagination, imagination directed by reason, "We have the power of constituting a single image out of a number of images."[13] In other places, the Stagirite also refers to the freedom we have in imagining.[14] It would seem to follow, therefore, that imagination under the direction of reason would be able to combine and correspondingly separate images and hence would be able to leave out (abstract) certain sensible features of physical objects grasped by the external senses. Thus, the resultant images of such abstraction would be that of uni and bidimensional qualityless objects, viz. true individual mathematical quantities. (Of course, this still leaves us with the question how nonsensible entities are able to be grasped by imagination which is the act of a sense faculty.)

Another area where imagination would seem to have a role in mathematics is in the generation and destruction of individual mathematicals. Since the mathematicals which are generated and destroyed must be individuals, for only individuals are generated and destroyed, they must be grasped by a sense power. But they cannot be actually grasped by the

external senses. Once again it would seem that it could be the imagination which would grasp such individuals. Now Aristotle does say that deliberative imagination is able to "constitute" a single image out of a number of images and correspondingly to separate one image into several; it would seem logical to conclude then that the generation of certain figures from other figures (e.g. the generation of a quadrilateral by joining two right triangles of the same size at their hypotenuse), and the corresponding destruction of others, would take place in imagination. No doubt the mathematician also usually constructs these figures and performs these various operations on tablets or something in order that they are visible to him. However, as we have noted, these sensible, spatially located, three-dimensional, mobile figures are not really the true, nonsensible, immobile, etc. continuous quantities of mathematics which he studies and manipulates (no more than the symbols he uses to designate numbers are really the numbers he manipulates), but only physical figures which more or less resemble them. In reference to the geometer and the figures he draws Aristotle says,

> [S]ome have held . . . that the geometer is uttering falsehood in stating that the line which he draws is a foot long or straight, when it is actually neither. The truth is that the geometer does not draw any conclusion from the being of the particular line of which he speaks, but from what his diagrams symbolize.[15]

The individual mathematical quantities he manipulates, as we said, would seem to demand location in the imagination. Again, however, I must hasten to add that the role of imagination in generation and corruption of the objects of mathematics is not

explicitly referred to by the Stagirite, though it does seem to follow logically from his expressed views.

If the individual mathematical objects are in imagination, it would follow that there is a certain amount of freedom on the part of the mathematician in regard to his objects. The fact that man, according to Aristotle, is able to direct his imagination by reason and is somewhat free in his imagining leads to this conclusion. Again, we are going beyond explicit statements of the Stagirite, but, if we are correct in locating individual mathematicals in this internal sense, it would seem to follow from what we have just seen that the mathematician would be free to construct in it, and then investigate, mathematicals (twenty-sided parallelograms for instance) which do not clearly correspond to anything found in the physical world. In other words, even though he doesn't say so, it would seem that for Aristotle the mathematician need not be limited to simply studying abstracted quantities which bear close resemblance to the quantities of things, but may treat quantities which he himself has devised in imagination and which have no one to one correspondence to any physical quantities. It is true that what the Stagirite does give as examples of geometrical objects are always rather elementary figures, circles, triangles, angles, etc. which could easily be gained by abstraction from similarly figured sensible things;[16] but I can see no reason why he would hold that the mathematician must limit himself to such easily abstractible entities. In fact, we might note in this connection,

that it is only the most general and basic elements, e.g. lines, planes, etc. that Aristotle explicitly mentions as obtained by abstraction.[17] The way is left open, apparently, for the mathematician using these elements to construct any figures he desires, any figures which are not self-contradictory of course (like square circles). The fact that the Stagirite refers to the necessity present in mathematics as of a hypothetical type may also indicate his recognition of the freedom of the mathematician to construct mathematicals as he chooses.[18] He says, "It is impossible, for instance, on a certain hypothesis that the triangle should have its angles equal to two right angles . . . ."[19] Some have interpreted this to mean that Aristotle holds that the mathematician is perfectly free to construct and define his figures (triangles for instance) any way he pleases — in other words that he is not bound to make them in one particular way or another.[20] (We will say more in the next chapter about the freedom of the mathematician in this regard and in his choice of hypotheses.)

In concluding this analysis of the role of imagination in mathematics, we should again point out the lack of explicit statements by Aristotle on the matters discussed. He never says that the quantities studied in mathematics are imaginable nor does he therefore give to imagination a special role in mathematical abstraction or in generation and destruction of mathematicals. Furthermore, he never explicitly refers to the freedom of the mathematician in regard to his choice of object, a freedom which seems to

be rooted in the freedom of imagination under the direction of reason; and in fact he mentions only those mathematicals which closely resemble quantified physical things. Nevertheless, the conclusions we have reached on these points do seem to follow from the Stagirite's expressed positions. That the imagination has a special role to play in mathematical abstraction and in generation and destruction of mathematicals seems to necessarily follow from the fact that he holds that individual mathematical quantities cannot be grasped by the external senses and yet as individuals must be grasped by a sense power. And if mathematicals are in imagination, since he teaches that imagination can be directed by reason it would seem to follow that the mathematician would have a great degree of freedom in constructing the objects he wishes to investigate.

It seems that once again the Stagirite has neglected to develop his own principles. His lack of reference to the role of imagination is apparently another omission in his philosophy of mathematics. Before categorically making such a statement however, we should remind ourselves that the omission of imagination by Aristotle may be deliberate. Keep in mind that we still have not answered this question — how can nonsensible objects like mathematicals be grasped by imagination which has sensible entities for its objects? It may very well be that Aristotle answered that they could not, and that this is the reason he does not consider imagination of relevance in mathematics. Again recall that the matter of mathematical quantities, of <u>individual</u> mathematicals, is al-

ways termed "intelligible" by him; he never calls it "imaginable." I will not attempt to push this analysis any further until Part II, for we will see there that St. Thomas Aquinas, who is much more explicit on this subject, provides a possible solution to this question.

FOOTNOTES

[1]Posterior Analytics, II, 19, 99b 36 - 100b 1.

[2]De Anima, III, 3. Memory, Aristotle states in the On Memory and Reminiscence, translated by J. I. Beare in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), 1, 450a 13-25, 1, 451a, 15-19, does have retained sensations for its object, but in addition it includes the perception of time. Moreover, memory is possible only after retention has taken place; in other words, it presupposes the retention of imagination.

[3]Some authors, particularly those inclined to read Aristotle through the eyes of St. Thomas, ignore this fact. Cf. for example, Mère St. Édouard, "La division . . .," 228 and Leroy, "Le savoir spéculatif," 303 ff. Maritain refers to the teaching of the "ancients" on this matter, The Degrees of Knowledge, translated from the French under the supervision of Gerald B. Phelan (4th ed.; New York, 1959), pp. 54-55. Needless to say, Aristotle is not one of the "ancients." Gabriel, "Matière intelligible . . .," 187, admits Aristotle does not have the doctrine and says one must "read between the lines" to find it.

[4]De Anima, translated with introduction and notes by R. D. Hicks (Cambridge, 1907), III, 3, 428b 13.

[5]In his On Memory and Reminiscence, Aristotle states that imagination is an "affection of the sensus communis." (1, 450a 12) In the De Somnis, translated by J. I. Beare in The Basic Works of Aristotle edited with an introduction by Richard McKeon (New York, 1941), on the other hand, he distinguishes between that power which is the controlling or judging sense faculty (apparently the sensus communis) and that which presents images (2, 460b 16-18; see also 3, 461b 18-31). Furthermore, he explicitly identifies the imaginative faculty with the sensitive faculty qua imaginative, though he does not say what this sensitive faculty is. (1, 459a 15-16) For some of the problems encountered in attempting to understand Aristotle's teaching on the imagination, see the article by Robert Roth, "The Aristotelian Use of Phantasia and Phantasma," New Scholasticism, XXXVII (July, 1963).

[6]De Anima, III, 8, 432a 9-14.

[7]De Anima, III, 4, distinguishes sense knowledge from

intellectual. See especially 429b 10-33. Also see Metaphysics, VII, 10, 1036a 1-12 and Posterior Analytics, I, 31, 87b 36-40; II, 19, 100a 15 - 100b 1. Also S. Cantin, "L'intelligence selon Aristote," Laval Théologique et Philosophique, IV (1948), 262-264.

[8]Metaphysics, VII, 10, 1036a 1-8, Owens' translation, p. 322 of The Doctrine of Being . . . .

[9]St. Thomas Aquinas interprets this intuition as imagination in In VII Metaphysics, L. 9, 1494-95.

[10]Pro, "Filosofía de la matemática . . .," points out that Aristotle never says, though he should have, that there are two faculties involved — one for grasping mathematical essences, one for individual mathematicals, 99.

[11]De Anima, III, 11, 434a 7-8.

[12]Nicomachean Ethics, VI, 1, 1139a 12-14.

[13]De Anima, III, 11, 434a 9, Hicks' translation.

[14]De Anima, III, 3, 427b 18-20.

[15]Posterior Analytics, I, 10, 76b 40 - 77a 2. See also Metaphysics, III, 2, 997b 33 - 998a 6.

[16]Heath points out both in . . . Greek Mathematics, I, p. 341 and Mathematics in Aristotle, p. 1, that Aristotle only refers to the most elementary geometrical figures.

[17]Thomas Greenwood, "Aristotle on Mathematical Constructibility," Thomist, XVII (1954), 89 and 93.

[18]Physics, II, 9, 200a 15-23; De Caelo, I, 12, 281b 3-7.

[19]De Caelo, I, 12, 281b 5-6.

[20]Greenwood, "Aristotle on Mathematical Constructibility," 91-93. Also Aristotle's discussion of mathematical principles in the Eudemian Ethics, translated by J. Solomon, Vol. IX, The Works of Aristotle translated into English under the editorship of W. D. Ross (Oxford, 1954), II, 6, 1222b 23 ff., indicates he holds there is some relativity in them.

CHAPTER IV

MATHEMATICS AS A SCIENCE

Mathematics as Scientific Knowledge

In order to understand mathematics as a science for Aristotle we should first discuss his general conception of science.

By science in the strictest sense Aristotle means knowledge of a necessary fact through its proper, proximate, cause.

> We suppose ourselves to possess unqualified scientific knowledge of a thing . . . when we think that we know the cause on which the fact depends, as the cause of that fact and of no other, and further, that the fact could not be other than it is.[1]

Clearly science in the unqualified sense means knowledge of a fact which cannot be other than it is (i.e. a necessary fact), knowledge in terms of its commensurately universal cause, i.e. the cause which is cause of this fact and of no other. More generally, science in a less strict sense refers to any certain knowledge of a fact through causes, even if these causes are not commensurately universal and even if they are not causes in reality of the fact but causes only of our knowledge of the fact.[2] In any case, whether it is taken in the strict sense or not, scientific knowledge for the Stagirite is knowledge of a necessary fact (and we will see that

there are different kinds of necessity) by means of causes (whether these causes be real causes of the fact or causes of only our knowledge of the fact).

Demonstration is the name Aristotle gives to the reasoning process productive of scientific knowledge.[3] Now in reasoning or syllogism (The Greek word "syllogism" means "reasoning.")[4] the middle term is the cause of the connection of the subject and the predicate in the conclusion. In the highest type of demonstration, where a fact is known for certain through its real proximate cause, the middle term will function not only as the cause of our knowledge of the fact, where the fact is the connection of subject and predicate in the conclusion (Every middle term has this function.), but will also express that cause which is really the cause of the fact.[5] Furthermore, in the premises of a demonstration the connection between the middle term and the subject of the syllogism and between the middle term and the predicate of the syllogism must be a necessary one. The reason for this is that the conclusion of a demonstration is necessary, i.e. the connection between its subject and predicate cannot be otherwise than it is. Since the conclusion derives its truth value from its premises, the propositions which are its premises must also be necessary, i.e. the connections between the subject and the middle term and between the predicate and the middle term cannot be otherwise than they are.[6] Aristotle explains that this requirement means that for a demonstration in the highest sense three conditions must be fulfilled as regards the relation between the

subject and the predicate of each of the premises: (1) the premises must involve universality, i.e. that in each of them their predicate must be true of every instance of the subject; (2) the relation between subject and predicate of each premise must be an essential relation, viz. one in which the predicate is an element in the essence of the subject, either as definition, as genus, or as differentia, or one in which the subject is an element in the definition of its predicate (viz. the predicate is a property of the subject), or one in which the subject is the necessary cause of the predicate;[7] finally, (3) the premises must have predicates each of which belongs to its subject and only to it (i.e. each must be commensurately universal with the subject), and hence the premises must be convertible.[8]

There are also other requirements which Aristotle lists for the premises of a demonstration, some of which would as a matter of fact be fulfilled if the above conditions are fulfilled, others of which would not. Since the conclusion is the effect of the premises, they must be true if it is to be true; furthermore,

> The premises must be primary and indemonstrable; [or reducible to such] otherwise they will require demonstration in order to be known [and this would involve one in an infinite regress] . . . . The premises must be the causes of the conclusion, better known than it and prior to it; its causes, since we possess scientific knowledge of a thing only when we know its causes; prior, in order to be causes; . . .

and we must know them better,

> [T]hat is, [be] more convinced of them — than their consequences, precisely because our knowledge of the latter is the effect of our knowledge of the premise.[9]

Now if the premises of a demonstration are to be commensurately universal (conditions [1] and [3] listed above) this would mean that the essential relation between the subject and predicate of each (condition [2]) must be such that the predicate is not the genus but either the definition, the difference, the property, or the necessary and commensurate effect of the subject. Of course the most complete way in which the predicate would be essentially related to the subject would be when it is the very essence of the subject, i.e. when it is the definition. Furthermore, that which is the most basic commensurately universal intrinsic cause of the inherence of an attribute in a subject is the very essence of that subject, for most fundamentally it is because of what something is that it has the properties it does. Therefore the highest type of demonstration (and by "highest" I mean a demonstration which sets forth the most basic commensurately universal cause and one whose premises embody the most complete essential relation between their terms) will be one which has the definition of the subject of the syllogism as the basis of the conclusion, i.e. one which contains the definition in its premises. The essence expressed in the definition will thus function in the demonstration as the cause, the middle term, of the necessary inherence of some attribute (the predicate of the syllogism) in the defined subject.[10]

We might also note that premises which contain the definition will also fulfill most of the other requirements which Aristotle lists for the premises of demonstration.

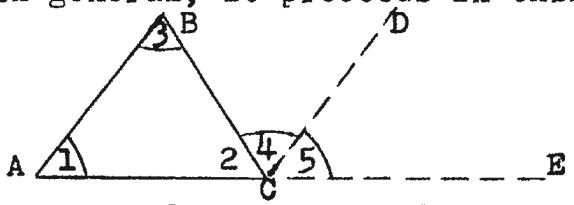
Certainly they are the causes of the conclusion inasmuch as they both cause our knowledge of the conclusion and express the real cause, the essence, which accounts for the inherence of the predicate in the subject which is stated in the conclusion. Since the premises contain the definition of the essence, and definition is indemonstrable (for there is no cause prior to it through which it could be demonstrated), they are primary and indemonstrable.[11] Inasmuch as the premises are causes of our knowledge of the conclusion and express the cause in reality of the conclusion they are prior to it. The premises are better known than the conclusion since what a thing is must be better known (at least in itself if not to us) than that which follows upon this.[12] So much for the nature of scientific knowledge in general. Let us now see in what sense mathematics is such knowledge.

We should realize from the start that Aristotle asserts that mathematics is a science in the highest sense of the word.[13] Probably the most highly developed science at his time was mathematics, a science of which he was well aware.[14] Indeed, in the Analytics the examples which he uses to illustrate demonstration are almost always taken from mathematics,[15] and furthermore even some of his very technical logical terms, e.g. axiom and figure, are borrowed from this field.[16]

Now to be a science in the highest sense mathematics must demonstrate through commensurately universal causes. It does just that, for Aristotle points out that "mathematics takes definitions, but never an accident, for its pre-

mises."[17] The mathematician, then, using the defined essence of a quantity as primary cause, demonstrates the necessary connection of an attribute with that quantity. For example, Aristotle states that the primary cause which is used to demonstrate that the interior angles of a triangle are equal to one straight angle is the very nature of the triangle itself.[18] To illustrate: the proof that the interior angles of a triangle equal one straight line, a proof which Aristotle refers to in his Metaphysics, is found in Euclid's Elements.[19] In general, it proceeds in this way:

Given: triangle ABC



To be proven: the interior angles of triangle ABC are equal to one straight line; or, angle 1 + angle 2 + angle 3 = a straight line.

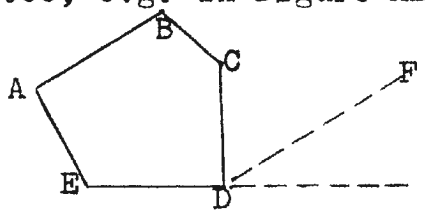
1. Extend side AC from C to E (shown in figure).
2. Draw line CD from point C parallel to AB (as shown).
3. Now angle 4 = angle 3 — this is known by a theorem dealing with parallel lines which has been demonstrated earlier, viz. if two parallels (AB and CD in the figure) are cut by a transverse line (BC) intersecting both, the alternate interior angles (viz. angles 3 and 4) are equal.
4. Also angle 1 = angle 5 — this also is known by a previously demonstrated theorem dealing with parallel lines, viz. if two parallels (AB and CD) are cut by a transversal (ACE in this case) the corresponding angles (1 and 5) are equal.
5. But angle 2 + angle 4 + angle 5 = one straight line as is

obvious from the figure.

6. Therefore, substituting angle 3 for its equal angle 4, and angle 1 for its equal angle 5, we obtain

7. Angle 1 + angle 2 + angle 3 = one straight line. Q.E.D.

The most primary cause of this conclusion is the very nature of the triangle itself, defined as a three-sided plane figure. It is true there are other causes of the conclusion, in particular the two previously proved theorems which are based on the nature of parallel lines and of the angles formed by lines which intersect them. However, these are not the most primary causes, for these theorems are able to be used in demonstrating the desired conclusion only because the triangle is the kind of figure it is, viz. one which has three sides such that a line (CD) can be drawn parallel to one side (AB) at the intersection of the other two (point C) making angles equal to the interior angles of the figure. If this parallel line could not be drawn so as to make the angles that it does the proof given above, using the theorem regarding parallel lines and their angles, would be impossible. But as I said this parallel line is able to be drawn to make the angles it does only because of the very nature of the triangle as having only three sides, any two of which intersect in a point not on the third side. Certainly a line can be drawn parallel to a side in a four or more sided figure too, e.g. in figure ABCDE line DF is drawn parallel to AB,



but it will not make angles equal to the interior angles of the given figure, and so it cannot be

used to prove the same theorem. Thus, to repeat, it is fundamentally the very nature of a triangle, defined as a three-sided plane figure, which causes it to have the demonstrated property. Likewise, it is not because a triangle is isosceles (having two sides equal) or equilateral (all three sides equal) or scalene (no sides equal) that it has this property but simply because it is a figure of three sides.[20]

A demonstration like this where the most primary commensurately universal cause of the inherence of a predicate in a subject is the very nature of the subject is scientific in the highest sense as we have shown above. And, as we noted, Aristotle says that mathematics always takes definition, i.e. the essence, as the basis of its demonstration. Of course, once conclusions have been drawn from these basic premises, the conclusions themselves can be used as the basis for further demonstrations and so on. In this way, the science of mathematics is built up.

Incidentally, since mathematical demonstrations are based upon the essence of mathematical quantities, and since this essence, according to our reasoning in Chapter II, must contain both form and common intelligible matter, it will follow that some mathematical demonstrations will be through formal cause and others through material cause. As we mentioned briefly in Chapter II, Aristotle does give a mathematical demonstration as an example of one which uses as its middle term the material cause.[21] The demonstration in question uses the <u>parts</u> of a right angle as the middle term; re-

call that the parts of a mathematical in relation to the whole are called its matter. If we can generalize from this example, it appears that for the Stagirite any proof in mathematics in which something is demonstrated of the whole through its parts is a proof through material cause. On the other hand, a demonstration through formal cause in mathematics would apparently be (Aristotle never explicitly says.) when something is demonstrated through the whole itself or through its shape — for example, a demonstration of some property of a triangle through the very shape (form) of a triangle.[22]

Since the bases of mathematical demonstrations are definitions (or are ultimately reducible to premises which contain definitions) they will fulfill most of the other requirements of the premises of a demonstration. As we noted above, they will be prior to and better known than the conclusion, primary and indemonstrable (at least reductively) and causes of the conclusion.

The one requirement for premises of a conclusion which is not mentioned above is that they must be true. In what sense will the propositions which are premises in mathematics be true? A proposition is true, Aristotle says, if it accurately states what is.[23] Mathematical propositions will be true, therefore, if they accurately state the way mathematicals are. However, we have seen that for Aristotle mathematical quantities as such do not exist as physical things, but actually <u>are</u> only in the mind of the mathematician (or perhaps more precisely in his imagination). It follows, then,

that mathematical propositions to be true must conform to the mathematicals known (or imagined) by the mathematician.[24] But does this mean then that any mathematical premise can be true, since the mathematician is not dealing with physical things already set before him but deals with things which to a great extent he himself causes to be? In other words, is the mathematician perfectly free, according to the Stagirite, to "create" any mathematical object or objects he wishes, define them as he pleases, and then using these definitions as premises, go on to demonstratively investigate their properties? Is every premise he states true simply because he makes it so? We spoke in the previous chapter about the freedom of the mathematician. Our question now concerning the truth of mathematical premises has to do with the limits, if any, of this freedom.

The Truth of the Premises of Mathematical Science, or, the Freedom of the Mathematician

Actually, not every premise the mathematician may state is necessarily true; Aristotle does set limits to his freedom in "creating" objects to be defined and further investigated. The mathematician is not free to deal with any quantity he might think of for Aristotle states that he must demonstrate that a particular quantity exists before he can define it (for there is no real definition of what does not exist),[25] and go on to demonstrate its attributes (what does not exist has no attributes).[26] I should add that the Stagirite does not say that the mathematician must demonstrate the existence of every quantity he deals with; this would,

of course, involve him in an infinite regress. No science, Aristotle says, proves the existence of the subject with which it deals.[27] The mathematician, therefore, does not demonstrate the existence of the most primary objects he treats: in geometry, for example, points and lines, in arithmetic, units. Rather he posits these as existing and defines them.[28] (A proposition which asserts the existence or nonexistence of something he calls an hypothesis; it is distinguished from a definition which gives the meaning but does not make such an assertion.)[29] Except, then, for those objects whose existence he supposes, the mathematician must demonstrate the existence of all the objects he deals with. Before he does this, Aristotle says, he has only a definition of all the terms (nominal definition) that he will use;he must demonstrate that operations, figures, numbers, etc. which correspond to these terms do exist.[30] We might recall in connection with this that it was pointed out in the last chapter that the Stagirite speaks only of the most basic elements of mathematics (viz. the ones expressed in hypotheses) as being attained directly by abstraction; the other more complex objects of mathematics must be demonstrated to exist.

Now the Stagirite does not explicitly say of what this demonstration of existence consists. According to the authorities on Greek mathematics, the actual practice used by the Greeks to demonstrate the existence of a particular mathematical quantity was to construct it.[31] Euclid, for instance, always constructed a particular mathematical notion before using it in a demonstration; for example, only after he had

constructed a square did he go on to treat it; only after he had constructed a perpendicular to a straight line did he use lines at right angles to one another.[32] Thomas Greenwood, who has investigated Greek mathematics, remarks, "If Aristotle had been against the practice of construction, which was current with the mathematicians of his time, he would have said so undoubtedly."[33] It seems safe to assume then that the Stagirite has construction in mind when he refers to the demonstration of existence used by mathematicians. This construction would apparently involve in the beginning the combining of the most basic abstracted elements — points, lines, surfaces, units — into various figures and numbers which in turn could be further combined into more and more complex objects. (Incidentally, in the area of arithmetic "construction" would no doubt be used analogously referring there to such operations as combining units, through addition and multiplication, or separating them, through subtraction and division.)

We have here given by Aristotle somewhat of a limitation on what the mathematician can and cannot treat. Except for his fundamental hypotheses he can really define and investigate only that which can be constructed. Entities which involve impossibility in their construction cannot be considered. It follows that as far as the premises and bases of his demonstrations are concerned they will be true, and hence real and not nominal definitions, if they are definitions of constructible mathematical quantities. Of course, once he has validly deduced conclusions from such premises these true

conclusions can themselves be used as premises in further demonstrations.

Actually, it is not at all clear that Aristotle considers the mathematician to be free to construct and hence treat any mathematical quantities that are not self-contradictory. Certainly the most basic abstracted elements, those whose existence is assumed, are so general as to be able to form any figure or number. Yet he never explicitly states that the mathematician has the freedom to construct these elements into any non-self-contradictory objects he pleases. In fact, as we noted earlier (Chapter III, p. 112) the only objects of geometry which he cites are those like circles, spheres, triangles, etc. which do closely resemble physical magnitudes. This may very well mean that mathematics as the Stagirite conceives it is restricted to constructing, and therefore treating, objects resembling quantified physical things.[34] However, Aristotle does not explicitly state this restriction either. Indeed, it might be argued that he could not have intended such a limitation of mathematics since numbers by their very nature are more abstract than magnitudes[35] and so are not able to be so closely bound to physical quantities.

In answer to this, it should be noted that among the Greeks arithmetic was closely tied to geometry and to actual physical magnitudes. In general number theory was treated by them in the framework of geometry.[36] From the time of the Pythagoreans on, numbers were often represented geometrically.[37] Euclid, for example, represents numbers by straight lines,

planes, squares, cubes, etc.[38] This is especially true of irrational numbers, e.g. the square root of two which could not be assigned a definite numerical value but could be represented by magnitudes.[39] Furthermore, the Greeks had no notion of imaginary numbers or of negative numbers, numbers which could hardly be said to correspond to numerical aspects of physical things. Instead the only numbers they used were the ordinary whole numbers and fractions, 1, 2, 3, $\frac{1}{2}$, $\frac{1}{4}$, etc. Interestingly enough, it is not until Diophantes (late third century A.D.) that we find any mathematical equations used which involve numbers raised to any power above three, the cube.[40] Apparently because there is no physical magnitude that has more than <u>three</u> dimensions the Greeks felt any higher power would be meaningless. The very terms they used in arithmetic, some of which are still in use today probably show more than anything else the geometrical framework in which this study was carried on. Our terms like square (A number is squared when it is multiplied by itself once.) and cube (A number is cubed when it is multiplied by itself once and this result in turn multiplied by the given number.) clearly indicate their geometrical origin. (Plato even refers to square and cube numbers as planes and solids respectively.)[41] Numbers were referred to by the Greek mathematicians as cubes, as squares, as oblong, triangular, polygonal, diagonal, as sides, as rectilinear, scalene, spherical, circular — all fundamentally <u>geometrical</u> terms.[42] A certain kind of proportion between numbers was called a geometrical proportion.[43] Various quadratic equations were solved geo-

metrically using the construction of figures.[44] Clearly, then, as we said, Greek arithmetic was closely tied to geometry and thence to physical magnitudes.

Since Aristotle too uses some of these geometrical terms in reference to numbers,[45] this may well indicate that he shares the views of his countrymen that arithmetic is closely related to geometry, that numbers somehow relate to physical magnitudes. Thus arithmetic too may be considered by the Stagirite to be restricted to constructing and hence treating objects in some way corresponding to physical things.[46] True, he in no place explicitly states that there is this restriction, on either geometry or arithmetic. Yet neither does he give an indication that he feels that the mathematician, either geometer or arithmetician, is free to construct or consider objects which do not in some way correspond to physical quantities. It seems reasonable to conclude, then, in the absence of statements to the contrary, that Aristotle in this respect is a man of his time, i.e. he considers the objects of mathematics to be idealized representations of actual physical quantities and the mathematician to be restricted to such objects.

Let us return now to the question of the truth of the premises of mathematical demonstrations. We have said above that these premises will be true if they are definitions of constructible mathematical objects. What about the hypotheses and the definitions of the most basic mathematical elements, which are after all the most fundamental premises used by the mathematician? In what sense, or under what con-

ditions, are they true? In what way, if any, is the mathematician restricted in his choice of them? Since hypotheses do not refer to objects which must be constructed, since they are propositions which simply assert that an object exists, it would seem that the mathematician would have almost complete freedom to set forth any hypothesis he pleases, so long as it is not self-contradictory, of course, like the existence of a dimensionless line.

We pointed out earlier (Chapter III, p. 113) that the Stagirite does refer to mathematical necessity as of a hypothetical kind. He states for example that on a certain hypothesis it is impossible that the interior angles of a triangle will equal two right angles; on a different hypothesis the value of the interior angles will be two right angles. Does this mean that either hypothesis is permissible? To generalize, does this mean that Aristotle feels that the mathematician can assert the existence of any non-self-contradictory quantity he pleases? He never says this. In fact, the only objects which the Stagirite explicitly says are assumed to exist by the mathematician are points, lines, and units,[47] very general elements which are very easily obtained by abstraction from physical things. Indeed, these elements are so general how could anything more basic in the genus quantity be abstracted?[48] As we said earlier, these elements could make up any figure or number. (Incidentally, the fact that these elements are so general and hence so easily abstracted may well be the reason why Aristotle says that little experience is needed in order to be-

come a mathematician.)[49] However, as we have already pointed out, the only objects composed of these elements which Aristotle mentions are, in the case of geometry, figures resembling physical quantities, and in the case of arithmetic real numbers, which at his time also corresponded to physical things and their quantitative features. It would seem to follow from this that the mathematician is limited in his hypotheses to positing the existence of elements which can make up mathematical quantities resembling (in the case of figures) or representing (in the case of numbers) physical quantities. There is only slight indication that Aristotle realizes that one could assume the existence of elements which would make up objects radically different from the Euclidean kind which more or less resemble real quantities (figures of non-Euclidean geometry, for example).[50] He does not pursue this point; he makes no explicit statement about any freedom allowed to the mathematician in his choice of hypotheses. This seems to be simply another case in his mathematical doctrine where the Stagirite has failed to develop his own premises. It appears he simply went along with the view of his time and considered mathematical objects to be limited to representation of physical quantities without noting that <u>his own principles did not demand such limitation of the scope of this science</u>.

Though it may be repetitious, let me dwell for a time on that last point. I can see no reason why the mathematician according to Aristotle's own principles would not be perfectly free in his choice of hypothesis to posit the

existence of any quantity or quantities. I cannot see why any hypothesis would not be true so long as it is not self-contradicting nor in conflict with another hypothesis. Why, for example, could the mathematical scientist not set forth hypotheses which express relations not between magnitudes or numbers but between symbols standing for any quantities, and then go on and deduce the various relations which could follow (as is done in algebra)? Why could the mathematician not set forth hypotheses asserting the existence of magnitudes which would not make up figures which resemble physical magnitudes (e.g. the magnitudes of non-Euclidean geometries), and of units (e.g. imaginary or minus units) not bound to make up real numbers representing physical quantities?

Except for the reference to the hypothetical character of mathematical demonstration, the only other places that might indicate that Aristotle realizes that mathematics need not be restricted to objects resembling physical quantities are his very obscure references to what he calls "universal mathematics."[51] This "universal mathematics" is a mathematical science more general than geometry and arithmetic. It seems to be a science which demonstrates attributes common to all quantities.[52] For example, Aristotle refers to a theorem about proportion which used to be demonstrated separately for numbers, lines, and solids, but now, he says,

> [T]he proposition is proved universally; for the property did not belong to the subjects qua lines or qua numbers, but qua manifesting this generic

> character which they are postulated as possessing universally.[53]

The "particular character which they are postulated as possessing universally" would evidently be quantity. This would indicate that universal mathematics has as its object quantity in general and its properties, not just magnitude or just number,[54] and so is something like algebra which uses symbols which can stand for either numbers or magnitudes.[55]

The fact that he has a notion of universal mathematics may indicate the partial realization by Aristotle that mathematics does not have to deal with quantities closely patterned on physical quantities. Actually, it is impossible to say for sure what universal mathematics is for him since he has so few references to it. One can wish that he had taken the time to develop this notion.

What can we conclude, then, about the truth of the hypotheses which the mathematician uses as his fundamental premises? I can see no reason according to the Stagirite's own principles why the mathematician would not be free to posit as existing and define any basic quantitative object or relations between objects. The only limitation would be that such an hypothesis and definition not be self-contradictory nor in conflict with other hypotheses and definitions. However, as we noted, the objects which Aristotle does say the mathematician posits as existing are those which are used to make up mathematical quantities which closely resemble or represent quantitative physical things. In other

words, the Stagirite seems to consciously or unconsciously limit the mathematician to setting forth hypotheses and treating objects corresponding to physical quantities. The only exceptions to this statement are found in his undeveloped notion of universal mathematics and in the slight indication he gives that he realizes that the premises of geometry are of a hypothetical character.

We have seen, then, how the premises used in mathematics whether they be hypotheses, definitions, or conclusions from prior premises, fulfill the requirements Aristotle lays down for premises of demonstration. We have spent most of our time attempting to determine how these premises fulfill the requirement of being true, a question which involved us in the related and very important question of Aristotle's position as to the freedom of the mathematician in regard to his premises. We still must see in what sense the object of mathematics is necessary, for science in the unqualified sense is of the necessary, that which cannot be other than it is.

Now we have already discussed Aristotle's view that the necessity present in mathematics is not of an absolute but a hypothetical kind. He states, for example, <u>if</u> there is a plane figure composed of three straight lines, it will necessarily have the sum of its interior angles equal to two right angles. But nothing absolutely demands that there be such a figure. Existing actually only in the thought of the mathematician, mathematical quantities are not eternal, they come to be and cease to be as they are considered by

him. Therefore, they are not necessary in the highest sense for that which is necessary in this sense is eternal.[56] However, they are necessary in the sense that once they do exist they are what they are and their attributes follow necessarily from their nature.

Thus, mathematics fulfills all the conditions the Stagirite set forth for scientific knowledge. It is science in the highest sense of the word. In concluding this consideration of mathematics as a science, it might serve as a good summary if we briefly outlined the procedure of the mathematician in developing his science as set forth by Aristotle. The mathematician begins with certain hypotheses, i.e. statements that a unit, line, point, exist, and he defines each of them. Such objects he apparently gains by an easy abstraction from physical things. He also begins with nominal definitions of various operations (e.g. squaring, intersection, perpendicular, etc.) and of figures, numbers, attributes, etc. Using the assumed objects, he then proceeds to perform various operations and construct his figures or numbers. Once he has shown by construction that a particular operation is possible and that a particular figure or number can exist he can give a real definition of them. After he has the real definitions of certain objects, the mathematician proceeds to demonstrate the properties of these objects. Once he has demonstrated the necessary connection between a property and its subject this conclusion can in turn be used as a premise for further demonstrations, and so on. Thus, the science is built up from the fundamental hypotheses.[57]

The "More Physical" Branches of Mathematics

Up till now when we have spoken of the mathematical sciences we have referred, except for the brief mention of universal mathematics, to its abstract branches — arithmetic, the study of quantitative plurality in the form of number, and geometry, the study of magnitudes. There are other branches of mathematics which are less abstract, less removed from physical things, branches which Aristotle refers to as "the more physical of the branches of mathematics."[58] These sciences, he says, are in a way the converse of the more abstract ones. Geometry, for example, studies lines abstracted from, apart from, the physical bodies in which they really exist; optics, on the other hand, studies lines not as separated from physical things but as present in them.

> These are in a way the converse of geometry. While geometry investigates physical lines but not qua physical, optics investigates mathematical lines, but qua physical, not qua mathematical.[59]

Optics might consider, for example, the linear properties of light.

A similar parallel could be drawn between harmonics and arithmetic. Both deal with the properties of number but arithmetic " . . . is not a science of properties qua inhering in a substratum . . .," while harmonics " . . . is a science of properties inhering in a substratum."[60] (The substratum in this case is sound.)[61] In addition to optics and harmonics, Aristotle mentions astronomy, which studies the phenomena of the heavens,[62] and mechanics, which con-

siders bodies acted on by other bodies in time and place, at rest and in motion,[63] as other "more physical" mathematical sciences.[64]

Though all of these sciences do deal with physical phenomena they are still mathematical sciences, for, Aristotle points out, they apply mathematical quantities and truths about them to physical phenomena. In order to do this they must, of course, consider the quantitative features of such phenomena.[65] Astronomy, for example, as the Stagirite describes it, studies the quantitative properties of the heavens. It seeks to determine the number of heavenly bodies[66] and the magnitude of their distance from each other and from the outermost sphere, and the time of their respective movements of rotation.[67] As for optics and harmonics, " . . . neither considers its object qua sight or qua voice, but qua lines and numbers; but the latter are attributes of the former."[68] And the Stagirite adds, "And mechanics too proceeds in the same way." In general, then, these more physical mathematical sciences apply or reinstate the mathematical quantities in natural things, the same quantities which arithmetic and geometry separate out by abstraction and consider in themselves. Hence the more physical mathematical sciences could be called applied mathematics as distinct from geometry and arithmetic, the abstract mathematical sciences.

As I said above, in order to apply mathematical quantities and truths about them to physical phenomena, the quantitative features of such phenomena must be dealt with.

Now we considered at length in Chapter I the meaning of quantity for Aristotle and pointed out that it is both magnitude (extension) and plurality divisible into quantitative parts. It is clearly such physical features then that are treated in the sciences which apply mathematical notions; for example, features like the number of heavenly bodies and their distances are studied in astronomy. But what about features like light and color studied in optics; sound and music, studied in harmonics, all of which are qualities; motion and time, studied in both astronomy and mechanics? None of these is actually a quantitative feature of physical things; yet they must be quantitative in some way if they are to be the object of applied mathematics. We briefly discussed in Chapter I the Stagirite's notion of <u>per accidens</u> or incidental quantity. We noted that he says that some entities which are not in themselves quantities can be considered quantitatively inasmuch as that in which they are is actually a quantity. In regards to motion and time he says that they are quanta incidentally.

> [T]hese also are called quanta of a sort and continuous because the things of which they are attributes are divisible. I mean . . . space . . . ; for because that is a quantum movement also is a quantum, and because this is a quantum time [which is the measure of movement][69] is one.[70]

In like manner, light and color are incidentally quantities inasmuch as the subject to which they belong, air or some body, is itself quantified.[71] Hence, all of these features, though in themselves not quantitative, can have quantitative principles and notions applied to them by the "more

physical" mathematical sciences.

Note that all of the abovementioned features are incidental continuous quantities, they are present in or related to magnitudes and hence geometrical notions are applicable to them. What about a quality like sound which Aristotle says is treated in harmonics, a science which applies arithmetical notions, notions of discrete quantities? In what sense is sound a discrete quantity? To answer this, we must turn to the Stagirite's notion of measure. As we saw in Chapter I, measure, he says, pertains properly to the category of quantity and involves determining the number of times a certain quantitative unit is present in the entity which is measured. However, he also says that measure may be extended to entities in categories other than quantity, and he specifically mentions various qualities (weight and sound) as measurable.[72] A nonquantitative feature is measured, he explains, by choosing something of the same kind as a unit and then determining how many times this unit is present in the entity to be measured. He cites, for example, the quarter-tone which is the unit of measure in music.[73] Measuring a particular note involves determining by some suitable instrument how many times this note is the unit. The measured entity can then be expressed by a number signifying some multiple of the chosen unit. Once translated into numbers these entities can be treated in an applied mathematical science which uses arithmetical notions. For example, harmonics seeks to determine which numerical ratios of one note to another produce harmonious sounds.

Thus nonquantitative entities can be treated by sciences which apply mathematical notions if these entities are incidental quantities or are expressible in quantitative terms through measurement. In fact, these two notions, incidental quantity and measure, which allow nonquantitative entities to be expressed and treated quantitatively would seem to be very fruitful in the study of physical phenomena, and indeed would seem to grant to the applied mathematical sciences a great scope in such a study. If one can generalize from what Aristotle says it would appear that any physical entity which is related to quantity or could be in some way measured and thus expressed quantitatively is able to be studied in applied mathematics. However, as many authors have noted, the Stagirite actually spends little time on these mathematical sciences, either in a philosophical discussion of their nature, or in the use of them in his own studies of nature.[74] His own studies of the physical world are almost exclusively of a nonmathematical character.

The probable reason for this neglect is his view that a physical explanation of things is more fundamental than a mathematical one. The Stagirite states explicitly that if there were no separate substances, which are the objects of metaphysics, then his science of physics would be the highest science.[75] It is clearly placed on a higher plane than mathematics, and the reason for this is not difficult to see. Physics, natural philosophy, explains things, he says, by their substantial principles, especially

in terms of their substantial form. Sciences which apply mathematics, on the other hand, consider physical things only qua quantitative, only in terms of features which are accidental. Just as substance is prior to and more basic than accident, so also a science explaining things by substantial principles will be prior to and more basic than one dealing with them in terms of accidental principles.[76]

Furthermore, as we have noted, mathematical entities actually exist only in the thought of the mathematician. Physical entities, on the contrary, have real existence of their own, as do the separate substances. But that which exists on its own is prior in being to that which exists only in another; therefore, in like manner, physics is prior to mathematics. Finally, since physical entities are prior even in knowledge to mathematical, inasmuch as the mathematical only come to be because of an abstraction based upon experience of the physical, so in this way too physics is prior to mathematics.

It is this priority of the science of physics over the science of mathematics which accounts, I think, for the Stagirite's relative lack of interest in mathematics, both in its abstract and its applied branches. This explains why mathematics is practically ignored by him in his physical investigations, even though, as we pointed out, it would seem to have a wide range of application due to his notions of incidental quantity and of measure.

To say that physics is prior to mathematics in studying physical things is not to deny that the branches

of applied mathematics are sciences in the strictest sense of the word. Aristotle affirms that these knowledges attain demonstrative knowledge through causes of the facts they investigate.[77] In fact one reason these areas of knowledge are considered by him to be branches of mathematics is that while their objects are physical phenomena, they gain scientific knowledge of these objects by means of mathematics. In all of these sciences,

> . . .[I]t is the business of the empirical observors to know the fact, of the mathematicians to know the reasoned fact; for the latter are in possession of the demonstrations giving the causes . . .[78]

For example, in astronomy, ". . . astronomical experience supplied the principles of astronomical science, for once the phenomena were adequately apprehended the demonstrations of astronomy were discovered. Similarly with any other art or science."[79] The demonstrations in astronomy came from mathematics, i.e. from the application of mathematics to the phenomena.[80] Likewise, to take an instance from optics, the physicist may know the fact of the rainbow, " . . . whereas to know the cause is for the student of optics, either simply as such or so far as he is concerned with the mathematics of it."[81] Similarly, acoustical harmonics is able to give a factual account of tones, but mathematical harmonics investigates their proximate and proper causes.[82]

In general, harmonics, optics, mechanics, and astronomy are related to geometry and/or arithmetic as subordinate to superior.

> . . .[T]here is another way too in which the fact and the reasoned fact differ, and that is when they

> are investigated respectively by different sciences. This occurs in the case of problems related to one another as subordinate and superior as when optical problems are subordinated to geometry, mechanical problems to solid geometry, harmonic problems to arithmetic, the data of observation to astronomy.[83]

That is, while the subordinate studies furnish the fact, the natural phenomena, to be accounted for, the superior sciences supply the scientific demonstration of these facts.

> . . .[T]he fact falls under a separate science (for the subject genus is separate), but the reasoned fact concerns the superior science, to which the attributes essentially belong.[84]

Aristotle is quite explicit on the branch of mathematics which furnishes the demonstrations for three of these sciences. Geometrical demonstrations apply to theorems in mechanics and optics, arithmetical demonstrations to harmonics.[85] In the case of astronomy he says only that the demonstrations are supplied by mathematics.[86]

Since mathematics can be used to find the cause of physical phenomena does this mean that mathematics can be for Aristotle a science of physical things? We have noted repeatedly his statements to the effect that mathematics is not a science of such things. However, this assertion needs qualification. Certainly mathematics can be said to be a science of physical things if by this is meant that it can study such things. We have just seen the areas of applied mathematics which are physical in this sense. On the other hand, if by " a science of physical things" is meant one that studies physical things <u>qua</u> physical then in this sense mathematics is not a science of physical things. As Aristotle has shown, the more physical branches

of mathematics since they apply mathematical notions to physical phenomena treat only the quantitative aspects of physical things. They study physical things but only qua quantitative or qua aspects in some way expressible in quantitative terms. This is not to deny that these aspects are studied as present in things. They certainly are in applied mathematics. Nevertheless, it is precisely in terms of their quantitative aspects, and aspects reducible to quantitative, and only in terms of these that physical things are treated in applied mathematics. A science of physical things qua physical must treat such things by including all their sensible features, and particularly by considering their substantial principles matter and form.[87] We might note in this connection that the Stagirite never calls the branches of applied mathematics physical sciences, nor does he ever say that they are intermediate sciences, located between the physical and the mathematical. They are mathematical, "more physical" than the purely abstract mathematical sciences of geometry and arithmetic, but mathematical nevertheless.[88]

The Speculative Character of the Mathematical Sciences

I will conclude this study of mathematical science in Aristotle with a discussion of the speculative and/or practical character of the branches of mathematics. I have referred above to the more physical mathematical sciences as applied sciences to distinguish them from purely abstract mathematical sciences; yet I did not call them practical

sciences. My reason for avoiding this latter terminology is that in Aristotle's view even these applied branches seem to be speculative sciences; he states categorically, "[M]athematics is theoretical . . . ."[89] The following will elaborate this point.

In general, sciences are divided by Aristotle into speculative or practical on the basis of their goal and "principles."[90] "[T]he end of theoretical knowledge is truth, while that of practical knowledge is action . . . ."[91] In regard to the goal or end it is important to distinguish the end of any specific science from the end of the scientist. For example, a man may know geometry in order to use it in surveying, in building, or in any other things of that kind. However, in itself geometry is not a science which treats of methods of surveying or building or any other way of <u>using</u> spatial magnitudes, but rather it seeks only to know the nature of magnitudes.[92] Any use made of it is incidental to this end. Aristotle is very definite on this point:

> [W]ith regard to theoretical knowledge . . . there is no other part of astronomy or physics or geometry except knowing and contemplating the nature of the things which are the subjects of these sciences; though nothing prevents them from being in an incidental way useful to us for much that we cannot do without.[93]

The only goal or aim of the speculative sciences is to contemplate their particular object, in the case of mathematics to know some kind of quantity and its attributes. These knowledges are not related to any other goal as a means to it, as knowing surveying is related to actually surveying land areas for example. In the area of knowledge the speculative

sciences have no final cause outside of themselves; they are an end. They are known just to be known, though they may be incidentally useful.[94]

We turn now to the distinction of sciences into speculative or practical according to their "principles". In the practical sciences, concerned as they are with doing or making, the "principle" of the sciences is within the subject who does or makes. According to the Stagirite the "principle" of the sciences concerned with doing is free choice; in the sciences concerned with making it is mind, art, or some faculty.[95] The speculative sciences, on the other hand, deal with things whose "principles" are in the things themselves, not in the possessors of the science. Aristotle holds that the science of nature, for example, since it is about that kind of thing which has its principles in itself, is a speculative science.[96] Speculative sciences take their beginning from principles within the thing they study, from the intrinsic causes of a thing. Practical sciences find in the doer or maker the causes, the principles, from which they proceed. The principles are not in the thing itself for it is something that must be done or made and so does not exist until the agent acts.[97]

It is clear that the more abstract mathematical sciences, geometry and arithmetic, are considered to be speculative by Aristotle. Indeed he lists mathematics as one of the three general kinds of speculative sciences.[98] In terms of end their speculative nature is evident. Their goal is truth, which is to say they have no end outside of themselves.

But in terms of their principles it is not so clear; they do not seem to be totally speculative. If we are correct in maintaining that construction is a very important part of mathematics then these sciences would be partly practical for the principles of construction must be within the mathematician himself. Construction like any making has its origin in the faculties, habits, and knowledge, in the art, of the one who constructs. It is true that once a particular mathematical quantity has been constructed it is investigated in terms of its own material and formal principles. Nevertheless, except for the elementary quantities whose existence is assumed, the existence of mathematicals must be demonstrated according to Aristotle, and we saw above that the common means used by the Greek mathematicians for such a demonstration was the construction of the objects in question. But, the principles and causes of a constructed object, before it is made, are not in the thing itself for it doesn't exist until it is made; they are in its maker. Thus in terms of their principles the abstract mathematical sciences would seem to be both speculative and practical for Aristotle. Insofar as most of their quantities are constructed they are practical, for the principles of making are in the mathematician himself. Insofar as these sciences investigate the constructed quantities in terms of their principles, they are speculative.

However, it should be pointed out that the Stagirite never explicitly states that these sciences are partly practical, no more than he ever explicitly refers to construction

as the means to demonstrate the existence of mathematical quantities. The closest he ever comes to affirming the practical side of mathematics is in one place in the Metaphysics in which he speaks of the mathematical arts,[99] and by art he means, "[A] state concerned with making, involving a true course of reasoning."[100] (underlining mine) Unfortunately it is not clear just what mathematical sciences he is referring to in his Metaphysics text. He could mean arithmetic and geometry and have in mind the making, i.e. the constructions, involved in them; but the text does not say.

Aside then from this one obscure passage there is no reference made by Aristotle to the practical character of mathematics. Perhaps the reason for this omission is that in terms of their end these sciences are purely speculative, for their ultimate goal is always truth not making or doing. The making which takes place in them is ordered to knowledge for the constructions are done ultimately so that a given quantity can be known. The making is for the sake of knowledge; the knowledge is not for the sake of making.[101] However, even this does not deny the fact that in terms of their principles they seem clearly to be practical as well as speculative, as we have explained.

As for the applied mathematical sciences, they appear to be totally speculative both in regard to their end and their principles. Astronomy, for example, which Aristotle explicitly calls theoretical (cf. the passage quoted on page 148), deals with the quantitative properties of the phenomena of the heavens. Even though the results of this science

are useful, in navigation for instance, the goal of astronomy is only to know the heavenly phenomena and not to use them. Moreover the "principles" of heavenly phenomena which account for their being the object of this applied mathematical science are the very quantitative features or features able to be treated quantitatively which they possess. The sources or principles from which the science of astronomy proceeds are attributes of the heavenly phenomena, not principles in the astronomer himself. Furthermore, the object of his science is not made by the astronomer. He applies mathematics to features he finds in the phenomena of the heavenly bodies. Therefore, from the point of view of both end and principles, astronomy is a speculative science. This same analysis can be applied to the other more physical mathematical sciences.

Optics may treat of the rainbow or linear properties of light. It does not seek to know how to cause rainbows or light, it does not make its objects, but seeks only to know them. Furthermore, the principles or sources from which this mathematical science begins are in the thing itself inasmuch as it contains quantitative features or features able to be treated quantitatively, not in the optician who merely studies and does not construct the features he finds. Similarly, the end of harmonics, the study of the numerical properties of sound, is simply to know causally the quantitative aspects of sound. To be sure harmonics can be put to use, in music for example, but this is not its goal. Moreover, the principles of the science of har-

monics are again the features of sound which can be studied quantitatively. So too mechanics does not seek to manipulate bodies but to know causally the quantitative aspects of the motion of bodies and their interactions. And the principles of the science of mechanics are not within the mechanic but are the accidents treated in terms of quantity possessed by the things themselves.

Clearly then, the more physical, the applied mathematical sciences, are in themselves speculative, even though they can be given a practical use. Their end is truth for they seek to know their particular subject mathematically, not to use it for some further end; and their principles are in that which they study, viz. their features which can be expressed quantitatively, rather than in the scientist who does the studying. There is no making involved in these sciences as there is in arithmetic and geometry.[102]

FOOTNOTES

[1]*Posterior Analytics*, I, 2, 71b 8-12.

[2]*Posterior Analytics*, I, 13, 78a 22-30; also see Owens, *The History of Ancient . . .*, p. 302; Smith, *The General Science . . .*, pp. 6-7.

[3]*Posterior Analytics*, I, 2, 71b 17-19.

[4]Owens, *The History of Ancient . . .*, p. 300.

[5]*Posterior Analytics*, II, 2. Edward D. Simmons, *The Scientific Art of Logic* (Milwaukee, 1961), pp. 270-271.

[6]*Posterior Analytics*, I, 4, 73a 21-24.

[7]I have omitted the third type of essential which Aristotle mentions, viz. "That is essential which is not predicated of a subject other than itself." (*Posterior Analytics*, I, 4, 73b 5-6) Substances are essential in this sense. This way of being essential is apparently added by the Stagirite only for the sake of completion and is not really a type of essential which relates to the premises of a demonstration. On this point, see Ross, *Aristotle*, pp. 48-49 and Simmons, *The Scientific Art . . .*, p. 264, footnote.

[8]These requirements for the premises of a demonstration are given by Aristotle in Chapter 4 of the first book of the *Posterior Analytics*. See also Ross's analysis of this material, *Aristotle*, pp. 48-49; Simmons, *The Scientific Art . . .*, pp. 263-264.

[9]This material on the requirements of the premises of a demonstration is from Chapter 2, Book I, *Posterior Analytics*. My quotations are all from that place. See also, Ross, *Aristotle*, p. 46; Simmons, *The Scientific Art . . .*, pp. 261-262.

[10]Aristotle states that "[T]he demonstration, in the essential sense, of any predicate is the proof of it as belonging to this first subject commensurately and universally." *Posterior Analytics*, I, 4, 74a 2-3) By "first subject" he means one to which the predicate belongs primarily and specifically. (I, 4, 73b 33 - a1). Certainly the most primary intrinsic cause of a predicate belonging to a subject commensurately and universally is the essence of that subject. On

pages 265-266 of his The Scientific Art . . ., Simmons sets forth the types of propter quid demonstrations. His explanation of each shows clearly that the first type which demonstrates the immediate property of a subject whose proximate cause is in principles intrinsic to the essence of its subject, is prior to types which demonstrate mediate properties. (See his footnote, p. 266.) Also see Owens, . . . Christian Metaphysics, p. 296.

[11]Posterior Analytics, II, 10, 94a 11-12; II, 3, 90b 23-31.

[12]Posterior Analytics, I, 2, 72a 28-30.

[13]Posterior Analytics, I,14, 79a 17-20:

"Of all the figures the most scientific is the first. Thus, it is the vehicle of the demonstrations of all the mathematical sciences, such as arithmetic, geometry and optics, ..."

[14]Almost any history of ancient science will substantiate this point. See Heath, . . . Greek Mathematics, I, Chapter 1. See also Ross, Aristotle, pp. 44-45 and van der Waerden, Science Awakening, Chapters 5 and 6.

[15]Many have pointed this out: Heath, . . . Greek Mathematics, p. 336; Ross, Aristotle, 47; Heath, Mathematics in Aristotle, p. 1; Greenwood, "The Characters of the . . .," 237.

[16]Ross, Aristotle, pp. 36-37. Greenwood, "The Characters of the . . .," 224.

[17]Posterior Analytics, I, 12, 78a 11-12.

[18]Aristotle states, "[T]he triangle in virtue of its own nature contains two right angles." (48a 36-37) See the whole discussion, Prior Analytics, translated by A. J. Jenkinson, Vol. I, The Works of Aristotle translated into English under the editorship of W.D. Ross (Oxford,1955), I, 35.

[19]Actually Aristotle does not give this proof in detail. However, he does clearly refer to the proof later found in Euclid. This reference is in Metaphysics, IX, 9, 1051a 24. See Heath commenting on this passage, pp. 216-217 of his Mathematics in Aristotle. For the proof itself I have relied on Heath, op. cit., pp. 26-30.

[20]Aristotle states this in the passages referred to in footnote 18. On the very nature of the triangle as the basis of this proof, cf. C. De Koninck, "Abstraction from matter: notes on St. Thomas's Prologue to the Physics," Laval Théologique et Philosophique, XIII (1957), 137-138.

[21]Posterior Analytics, II, 11, 94a 28-35.

[22]Aristotle states that there is no demonstration through final cause in mathematics, Metaphysics, III, 2, 996a 21-36. Apparently there is no demonstration through efficient cause either; at least he never refers to any. However, I think these statements need qualification. Certainly mathematical quantities do have an efficient cause; one mathematical may serve as agent in the production of another (e.g. the line which bisects another line is the agent cause of the two new lines). Ultimately, of course, the efficient cause of mathematical figures and numbers is the mathematician himself. (Cf. Metaphysics, IX, 9, 1051a 30: " . . . [T]he potentially existing constructions are discovered by being brought to actuality; the reason is that the geometer's thinking is an actuality . . . .") Furthermore, mathematicals have a final cause; the mathematician has some end in mind when he makes and manipulates his quantities (e.g. he draws one line to bisect a given line for the purpose [final cause] of obtaining two lines equal in length). Now these efficient and final causes do enter into some mathematical "demonstrations," namely those which Aristotle calls demonstrations of existence. I am referring of course to the constructions which show that various mathematical figures and numbers exist (See above pp. 128-29.). To take a simple example of such "demonstrations": in order to show (demonstrate) that there is such a figure as a right triangle Euclid first constructs a perpendicular to a given line. The efficient cause of the right angle is the line which is drawn perpendicular; the final cause is the triangle whose existence is to be demonstrated. Clearly both causes are integral to this demonstration. However, once the existence of the particular figure or number has been demonstrated by construction, it is investigated demonstratively solely in terms of its intrinsic causes (matter and form) and no longer in terms of the extrinsic causes which account for its production. In other words, once the existence of a particular mathematical quantity has been demonstrated, there is no demonstration of its properties through efficient or final causes.

[23]Metaphysics, IX, 10, 1051b 5-9.

[24]A. E. Taylor in Aristotle (New York, n.d.), p. 33 seems to feel that since mathematical propositions "are not concerned with actual relations between sensible data" they "are not strictly true." I can see no reason for limiting true propositions to those which are concerned with actual sensible objects. It seems to me that truth would have an analogous meaning depending upon the nature of the particular objects in question, as sensible, or mental (logic and mathematical objects) or really existing and purely intelligible (objects of metaphysics).

[25]Posterior Analytics, II, 7, 92b 4-7; II, 10, 93b

33-35. See Apostle, . . . Philosophy of Mathematics, p. 38.

[26]Posterior Analytics, I, 10, 76b 5-10:

"Both the existence and the meaning of the subjects are assumed by these sciences; but of their essential attributes only the meaning is assumed. For example arithmetic assumes the meaning of odd and even, square and cube, geometry that of incommensurable, or of deflection or verging of lines, whereas the existence of these attributes is demonstrated by means of the axioms and from previous conclusions as premisses. [sic]"

II, 7, 92b 12-17:

"[W]e hold that it is by demonstration that the being of everything must be proved — . . . Hence the being of anything as fact is matter for demonstration; and this is the actual procedure of the sciences, for the geometer assumes the meaning of the word triangle, but that it is possessed of some attribute he proves."

and I, 10, 76a 33-37.

[27]Posterior Analytics, I, 10, 76b 3-23.

[28]Posterior Analytics, I, 10, 76b 3-11:

"Also peculiar to a science are the subjects the existence as well as the meaning of which it assumes, and the essential attributes of which it investigates, e.g. in arithmetic units, in geometry points and lines. Both the existence and the meaning of the subjects are assumed by these sciences; but of their essential attributes only the meaning is assumed. For example arithmetic assumes the meaning of odd and even, square and cube, geometry that of incommensurable, or of deflection or verging of lines, whereas the existence of these attributes is demonstrated by means of axioms and from previous conclusions as premisses. [sic]"

II, 8, 93b 21-25:

"Now while some things have a cause distinct from themselves, others have not. Hence it is evident that there are essential natures which are immediate, that is, are basic premisses [sic]; and of these not only that they are but also what they are must be assumed or revealed in some other way. This too is the actual procedure of the arithmetician, who assumes both the nature and the existence of unit."

and I, 10, 76a 31-36.

[29]Posterior Analytics, I, 2, 72a 19-24.

"If a thesis assumes one part or the other of an enunciation, i.e. asserts either the existence or the non-existence of a subject, it is a hypothesis; if it does not so assert, it is a definition. Definition is a 'thesis' or a 'laying something down', since the arithmetician lays it down that to be a unit is to be quantitatively indivisible; but it is not a hypothesis, for to define what a unit is is not the same as to affirm its existence."

All of the following agree in this interpretation of Aristotle: Ross, Aristotle, p. 47; Heath, Mathematics in Aristotle, pp. 50-57; Gabriel, "Matière intelligible . . .," 178; Greenwood, Etudes sur . . ., p. 56 and "The Characters of the . . . .," 243-244.

[30]Posterior Analytics, I, 10, 76b 3-16:

"Also peculiar to a science are the subjects the existence as well as the meaning of which it assumes, and the essential attributes of which it investigates, e.g. in arithmetic units, in geometry points and lines. Both the existence and the meaning of the subjects are assumed by these sciences; but of their essential attributes only the meaning is assumed. For example arithmetic assumes the meaning of odd and even, square and cube, geometry that of incommensurable, or of deflection or of verging of lines, whereas the existence of these attributes is demonstrated by means of axioms and from previous conclusions as premisses.[sic] Astronomy too proceeds in the same way. For indeed every demonstrative science has three elements: (1) that which it posits, the subject genus whose essential attributes it examines; (2) the so-called axioms, which are primary premisses [sic] of its demonstration; (3) the attributes, the meaning of which it assumes."

On this point see also Ross, Aristotle, p. 47; Apostle, . . . Philosophy of Mathematics, pp. 42-45; and Heath, Mathematics in Aristotle, pp. 70-71; Greenwood, "The Characters of the . . . ," 244.

[31]Heath, . . . Greek Mathematics, I, pp. 337 and 377; Greenwood, "Aristotle on Mathematical Constructibility," 89-93; Apostle, . . . Philosophy of Mathematics, p. 42.

[32]Greenwood, "Aristotle on Mathematical Constructibility," 93; Heath, . . . Greek Mathematics, pp. 337 and 377; De Koninck, "Abstraction from matter . . .," 135-137.

[33]Greenwood, "Aristotle on Mathematical Constructibility," 93. Aristotle does speak of construction in mathematics, e.g. De Caelo, I, 10, 280a 1 ff., but he never says that this is the way to demonstrate existence in mathematics.

[34]Greenwood, "Aristotle on Mathematical Constructibility," 93-94, and "The Characters of the . . .," 244,

seems to hold this position.

[35]Posterior Analytics, I, 27, 87a 31-37; Metaphysics, XIII, 3, 1078a 8-12.

[36]Heath says,

"With rare exceptions . . . the theory of numbers was only treated in connexion with geometry, and for that reason only the geometrical form of proof was used, whether the figures took the form of dots marking out squares, triangles, gnomons, etc. (as with the early Pythagoreans), or of straight lines (as in Euclid VII -IX) . . . ."
(. . . Greek Mathematics, I, p. 16)

Heath also points out that even problems which we would call algebraic were only solved geometrically by the Greeks. (Mathematics in Aristotle, p. 223, also . . . Greek Mathematics, I, pp. 379 ff. See also his explanation of "geometrical algebra," pp. 150-154.) On this point, see also van der Waerden, Science Awakening, pp. 118-126.

See also M. R. Cohen and I. E. Drabkin, A Source Book in Greek Science (Cambridge, 1958), p. 1 and p. 14, n.1.

[37]Heath, . . . Greek Mathematics, I, p. 76 ff.

[38]Heath, . . . Greek Mathematics, I, pp. 16, 98 and 379 ff.; Mathematics in Aristotle, p. 222.

[39]The square root of two would be represented simply by drawing a square of sides one and one whose diagonal would then be the square root of two. Many authorities feel that it was the discovery of the irrational that turned the Greeks in the direction of geometry and accounted for the "geometrizing" of number. See, for example, Marshall Clagett, Greek Science in Antiquity (New York, 1955), p. 57 and Cohen and Drabkin, A Source Book in Greek Science, p. 1 and p. 14, n. 1.

[40]Cohen and Drabkin, A Source Book in Greek Science, p. 25. Collingwood, "Is 'Physical Knowledge' Limited by Its Quantitative Approach to Reality?" in The Nature of Physical Knowledge edited by L. W. Friedrich (London, 1961), pp. 27-28.

[41]The reference to Plato is in Heath, . . . Greek Mathematics, I, p. 89.

[42]All these expressions can be found between pages 76 and 117 in Heath, . . . Greek Mathematics, I.

[43]Heath, . . . Greek Mathematics, I, p. 85.

[44]Cf. Heath and van der Waerden, pages cited in

footnote 36.

[45]At Physics, III, 4, 203a 13-15, Aristotle speaks of gnomons (which are figures) being put around numbers to make them various figures. See Heath's comment on this passage in Mathematics in Aristotle, pp. 101-102. At Posterior Analytics, I, 12, 78a 4 and Nicomachean Ethics, V, 3, 1131b 12-15, he speaks of the geometrical proportion. In the Metaphysics, V, 14, 1020b 3-6, Aristotle refers to number in one or more dimensions, ". . . numbers which are composite and not of one dimension only, viz. those of which the plane and the solid are copies." [underlining mine] and of other similar features of numbers which he calls their "qualities." See Heath's comments on this, Mathematics in Aristotle, p. 209.

[46]I do not mean to imply by this that Aristotle denies the specific distinction between arithmetic and geometry, between number and magnitude. Just the opposite. For instance, he criticizes the Pythagoreans for turning units into magnitudes. Furthermore, the Stagirite's references to number in geometrical terms are few. Nevertheless, even though he does assert the specific difference between the objects of arithmetic and of geometry, there is no indication that this leads him to disagree with his contemporaries who consider number in a geometrical context as representative of magnitudes. Numbers certainly are not magnitudes; they cannot be reduced to magnitudes; but still they can represent (Aristotle calls them copies in text of previous footnote.) magnitudes. (As 3^2 can represent the square of sides 3, as It seems to me that if he had disagreed with this geometrical treating of numbers Aristotle would have said so. □

[47]See the texts cited in footnote 28.

[48]Metaphysics, XI, 2, 1060b 12-17; XIV, 3, 1090b 5-6.

[49]Nicomachean Ethics, VI, 8, 1142a 16-19.

[50]Heath, Mathematics in Aristotle, p. 101, makes the following remarks about Aristotle's comment on the arbitrary character of the geometer's definition of a straight line:

"It is as if he had had a sort of prophetic idea of some geometry based on other than Euclidean principles, such as the modern 'non-Euclidean' geometries . . . . It is not possible that Aristotle could consciously have conceived such an idea as Riemann's [one of the founders of a non-Euclidean geometry]; but we can imagine him asking himself the question, 'If the angles of a triangle were together not equal to two right angles, what sort of a thing would a straight line have to be?'"

[51]Places where he explicitly refers to universal math-

ematics are: Metaphysics, VI, 1, 1026a 27; XI, 7, 1064b 8-9. He talks about universal propositions or theorems of mathematics at: Metaphysics, XIII, 2, 1077a 9-10; XIII, 3, 1077b 17-20; XIV, 1, 1087b 15-18 and Posterior Analytics, I, 5, 74a 17-25.

[52]Some have suggested that by this term he means a general type of mathematics which investigates the most general axioms such as "if equals are subtracted from equals the remainders are equals," "the whole is greater than any part of it," etc. (Apostle, . . . Philosophy of Mathematics, p. 66) However, Aristotle states in his Metaphysics that it is first philosophy, not mathematics, which studies axioms such as these common to all quantities. (Metaphysics, XI, 4, 1061b 17-28.) Apostle does seem to realize this, op. cit., p. 34, and yet still says they are treated by universal mathematics, p. 66.

[53]Posterior Analytics, I, 5, 74a 23-25, combining Heath's translation (p. 41 of Mathematics in Aristotle) with that of Mure in The Basic Works of Aristotle.

[54]In additionto the preceeding text, other places where Aristotle indicates that universal mathematics deals with quantity in general and not with magnitude or number are: Metaphysics, XIII, 5, 1077b 17-20 and Posterior Analytics, II, 17, 99a 8-11. It is Heath's position that the object of universal mathematics is quantity in general. See his commentary on these various passages in Mathematics in Aristotle, pp. 41-44, 223.

[55]In this connection, Heath's remarks on the nature of Aristotle's universal mathematics are very interesting. (Mathematics in Aristotle, p. 223)

"Perhaps the best analogy for the sort of 'general' or 'universal' mathematics which Aristotle seems to have in mind is our algebra which supplements arithmetical notions and operations by the use of general symbols for things known or unknown, and for operations. (Even so, the things denoted by the symbols are generally spoken of as 'quantities.')"

I might mention in this connection that in the places where he does use symbolic notation to express quantitative relations the symbols stand not for pure abstract quantities but for quantitative aspects of physical phenomena. (See for example Physics, IV, 8, 215b 1-11; Heath's explanation, p. 116 of his Mathematics in Aristotle. See also De Caelo, III, 2, 301a 26 - b1. For others see Heath, . . . Greek Mathematics, I, pp. 344-346 and Mathematics in Aristotle, pp. 116-119, 125-128, 142-146, 151-152, 165-167.) In other words, the symbols are not functioning in the area of universal mathematics for they are not used to represent purely abstract quantities and their relations. Rather, as limited to rep-

resenting quantities and their relations found in physical things, they are in the area of applied mathematics. (Heath, . . . Greek Mathematics, I, 344-46, lists these examples in the applied science of mechanics. See below, footnote 63.)

[56]Nicomachean Ethics, VI, 3, 1139b 23-24.

"[T]hings that are of necessity in the unqualified sense are all eternal."

See also On the Parts of Animals, I, 1, 639b 24-27 and Metaphysics, V, 5.

[57]The procedure of the mathematician given by Aristotle and outlined by me in the text is substantially that followed by Euclid in his Elements. Heath makes this point in his . . . Greek Mathematics, p. 337. Also check his outline of Euclid's Elements, op. cit., p. 373 and ff. Needless to say, the fact that their procedures are generally the same does not necessarily mean that the Stagirite commits himself to a geometry of the type devised by Euclid.

[58]Physics, II, 2, 194a 7.

[59]Physics, II, 2, 194a 8-11.

[60]Posterior Analytics, I, 27, 87a 32-33.

[61]Metaphysics, XIII, 3, 1078a 15. See also Heath, . . . Greek Mathematics, 13 and 17 of Vol. I.

[62]Aristotle's many references to astronomy make clear that this is its object. See for example Physics, II, 2, 193b 25-31; De Caelo, II, 10, 291a 30-35; Metaphysics, III, 2, 997b 16-18, XII, 8, 1073b 1-14.

[63]The spurious work, the Mechanics, translated by E.S. Forster, Vol. VI of The Works of Aristotle translated into English under the editorship of W.D. Ross (Oxford, 1952), indicates the type of thing studied in this field during Aristotle's time. See also Heath, . . . Greek Mathematics, I, 18. I think that many, if not all, of the times that Aristotle uses symbolic notation dealing with bodies in time and space acted on by other bodies (e.g. Physics, VII, 5, 249b 30 -250a 7), he is in the area of mechanics. Heath lists such places in mechanics, cf. . . . Greek Mathematics, I, 344-46.

[64]Some of the places where he refers to two or more of these branches of mathematics: Physics, II, 2, 194a 7-11; Metaphysics, III, 2, 997b 16-21, XIII, 3, 1078a 14-17. In addition to the four that Aristotle explicitly mentions, the ancients also had applied mathematical sciences of geodesy, geometry applied to actual land surfaces and

volumes, and logistics, properties of numbers applied to sensible objects. On this point, see Heath, . . . Greek Mathematics, I, pp. 13-18.

[65]Apostle, . . . Philosophy of Mathematics, pp. 131-132.

[66]Metaphysics, XII, 8, 1073b 2-14.

[67]De Caelo, II, 10, 291a 30-35, b 8-10.

[68]Metaphysics, XIII, 3, 1078a 15-17.

[69]Physics, IV, 11, 220a 24-26.

[70]Metaphysics, V, 13, 1020a 30-33.

[71]Metaphysics, V, 13, 1020a 25-28.

[72]Metaphysics, X, 1, 1052b 20 ff. See the entire chapter. For a discussion of the measurement and mathematical study of qualities, see the excellent article by Mark Heath, "Can Qualities Be Measured?"

[73]Metaphysics, X, 1, 1053a 13.

[74]Mansion, "La physique aristotélicienne . . .," for one says this is the biggest defect in Aristotle's physical science, his neglect of applied mathematics, 16-26.

[75]Metaphysics, VI, 1, 1026a 28-31.

"[I]f there is no substance other than those which are formed by nature, natural science will be the first science; but if there is an immovable substance, the science of this must be prior and must be first philosophy, ..."

[76]Aristotle states, "[E]ach science is called better or worse in virtue of its proper object." Metaphysics, XI, 7, 1064b 5-6. He says the same in Metaphysics, VI, 1, 1026a 20 ff. See also Owens, "The Aristotelian Conception . . .," 211 and Maritain's discussion of the priority of an ontological to a mathematical treatment of nature in his Philosophy of Nature (New York, 1951), Chapter III, pp. 73 ff.

[77]Owens discusses the propter quid character of the intermediate sciences for Aristotle in "Our Knowledge of Nature," 68, 69 and 72.

[78]Posterior Analytics, I, 13, 79a 2-4. See also Apostle's discussion of this, . . . Philosophy of Mathematics, pp. 135-137.

[79]Prior Analytics, I, 30, 46a 19-22. See Owens'

discussion of this text, "Our Knowledge of Nature," 68,69.

[80]On the Parts of Animals, I, 1, 639b 8-11. Indeed, in some cases Aristotle appears to equate astronomers with mathematicians. Cf. De Caelo, II, 10, 291a 30-35, b 8-10.

[81]Posterior Analytics, I, 13, 79a 11-13. (From Heath's translation in Mathematics in Aristotle, p. 59)

[82]Posterior Analytics, I, 13, 78b 40- 79a 4.

[83]Posterior Analytics, I, 13, 78b 33 - 79a 1.

[84]Posterior Analytics, I, 9, 76a 12-14.

[85]Posterior Analytics, I, 7, 75b 14-16; I, 9, 76a 23-26.

[86]De Caelo, II, 10, 291b 8-10; II, 14, 297a 4-5; On the Parts of Animals, I, 1, 639b 8-9.

[87]De Anima, I, 1, 403b 7-16:

"Which, then, among these is entitled to be regarded as the genuine physicist? The one who confines himself to the material, or the one who restricts himself to the formulable essence alone? Is it not rather the one who combines both in a single formula? . . . The physicist is he who concerns himself with all the properties active and passive of bodies or materials thus or thus defined; . . . ."

Metaphysics, VI, 1, 1025b 19 - 1026a 6. Also see Owens, The Doctrine of Being . . ., pp. 239-240 and "Our Knowledge of Nature," 62 and 81, where he discusses the difference between the mathematical and physical treatment of physical things, and Maritain, Philosophy of Nature, Chapter III, § 14.

[88]Owens, "Our Knowledge of Nature," 67, and Mansion, "La physique aristotélicienne . . .," 17, point out that Aristotle unequivocally calls these sciences mathematical. Those authors who persist in interpreting Aristotle as a Thomist ignore this point and attribute to him Thomas's doctrine that these sciences are intermediate. (e.g. Leroy, "Le savoir spéculatif," 305 and Gabriel, "Matière intelligible . . .," 182)

[89]Metaphysics, XI, 7, 1064a 31; 1064b 1-2:

"[T]here are three kinds of theoretical sciences — physics, mathematics, theology."

Aristotle says the same thing at VI, 1, 1026a 7-20. Owens, "The Aristotelian Conception . . .," states, "All the mathematical sciences, for instance, came under the designation

'speculative' or 'theoretical' for the Stagirite." [underlining mine]

[90]Owens, The Doctrine of Being . . ., p. 347, note 20 lists all the places where Aristotle refers to this division of the sciences.

[91]Metaphysics, II, 1, 993b 20-31.

[92]Nicomachean Ethics, I, 7, 1098a 28-32.

[93]Eudemian Ethics, I, 5, 1216b 12-16. In the Protrepticus too Aristotle distinguishes between the theoretical science of geometry and the applied disciplines such as surveying which make use of it. Werner Jaeger has this text in Aristotle: Fundamentals of the History of the Development of His Thought, translated by Richard Robinson (2d ed.; Oxford, 1948), p. 71.

[94]Owens, The History of Ancient . . ., p. 295.

"The goal of the speculative sciences consists in knowledge of truth. That is their good, their purpose. The good of the other two kinds of science [practical and productive], however, lies in something over and above the knowledge that they entail. Their purpose, their good, is respectively action or a product."

[95]Metaphysics, VI, 1, 1025b 22-25.

[96]Metaphysics, VI, 1, 1025b 19-28.

[97]Owens, A History of Ancient . . ., pp. 294-295. See also his "The Aristotelian Conception . . .," 206.

[98]See texts in footnote 89.

[99]Metaphysics, I, 1, 981b 19-24.

"[W]hen all such inventions were already established, the sciences which do not aim at giving pleasure or at the necessities of life were discovered, and first in the places where men first began to have leisure. This is why the mathematical arts were founded in Egypt; for the priestly caste was allowed to be at leisure."

For a discussion of this, see Robert Smith, "The Liberal Arts," Laval Théologique et Philosophique, II (1946), 79 ff. A. Nemetz, "Logic and the Division of the Sciences in Aristotle and Thomas Aquinas," Modern Schoolman, XXXIII (1956) says that a speculative art is a self-contradiction for Aristotle, 103. If this were so, then Aristotle would be guilty of just such a contradiction for he calls mathematics both speculative (footnote 89) and art (above text). However, why such

a notion should be self-contradictory for the Stagirite, I cannot see. A speculative art is simply a liberal art.

[100] Nicomachean Ethics, VI, 4, 1040a 20-21.

[101] On this point, see James V. Mullaney, "The Liberal Arts in the Aristotelian Thomist Scheme of Knowledge," Thomist, XIX (1956), 491 ff.

[102] Again see Mullaney's discussion of the purely speculative character of the applied mathematical sciences, ibid., 494-96.

CHAPTER V

CONCLUSION OF PART I

I am not at this time going to undertake a comprehensive summary of the Stagirite's philosophy of mathematics for in Part II we will be returning again and again to his doctrines comparing and contrasting them with those of St. Thomas Aquinas. Any summary can wait until the completion of Part II.

Though we find in Aristotle what is probably the most comprehensive treatment of his time of the nature of mathematics (in regard to the nature of its object, its character as theoretical knowledge which is scientific in the highest sense, and its various branches), we also find that he has a significant number of omissions, of points left undeveloped, of loose ends left loose. Of course, a man cannot develop in detail all areas of knowledge in his lifetime and certainly he will devote himself to those fields in which he is more interested and which he feels are the most important. Since, as we noted above, Aristotle considered both metaphysics and physics to be superior to mathematics, it is no wonder that the latter was developed the least by him.

It might be well in concluding this first part to briefly list some of the crucial points omitted or left

undeveloped by the Stagirite. In the very first chapter, we noted that the distinction which he clearly makes between quantitative and nonquantitative units and numbers is never applied to the area of mathematics. Instead, he often talks as if arithmetic deals with any unit and number and not just with quantitative ones. Also in the first chapter we noted that he fails to enunciate any relation between quantity and matter, though the description of quantity which he gives would seem to demand such a relation. This latter point became very important in Chapter II where we sought to determine what precisely is Aristotle's doctrine on the constitution of the essence of mathematical quantities. We asked — do mathematical essences according to the Stagirite consist of form only or of form and common intelligible matter? We saw that he **never** clearly answers this question though on the basis of his teaching concerning the nature of the accident quantity, it would seem that matter must be present in abstracted quantities; and yet many of his texts seem to say otherwise. Of course, the whole question of the meaning of his notion of intelligible matter is pertinent here. As we pointed out, his explanation of this "matter" is very brief and incomplete. Also closely related to the question of the ingredients of the essence of mathematical quantities is the very basic question — also unanswered by the Stagirite — why is abstraction possible at all? Why can certain features of physical things be left aside while other features are studied by the mathematician in isolation, or to put it another way,

what features can be left aside and what cannot by the mathematician's act of abstraction and why? As we noted in Chapter II, if Aristotle had answered this most fundamental question it would have been of great help in determining the necessary ingredients of the mathematical object.

Also omitted by the Stagirite is any discussion of the role of imagination in mathematics. It seems that much could be said about this power as the location of individual mathematicals — yet nothing is. Closely related to this is the question of the freedom of the mathematician as regards his choice of object and of hypotheses for his science. This freedom is not discussed, nor is the practical side of mathematics mentioned, except for one obscure reference to the mathematical arts. Perhaps the most important unresolved question concerning the freedom of the mathematician is this — is mathematics limited by Aristotle to a consideration of quantities corresponding to physical quantities? It is only such quantities that he cites as objects of mathematics and yet there seems to be no reason, according to his own principles, why such a restriction should be placed on the mathematician. The Stagirite's notion of universal mathematics, plus his references to the hypothetical character of mathematical necessity, may indicate that he has some inkling of the tremendous freedom open to the mathematician. Yet, as I said, nothing is really developed along this line. In fact, even though its doctrinal basis seems present already in the Stagirite's mathemat-

ical teachings, it took many centuries before this freedom was realized.

These, then, seem to me to be the main points omitted or at least undeveloped by Aristotle in his philosophy of mathematics. We turn now to Part II of this paper in which we will set forth St. Thomas Aquinas's mathematical doctrines. We will take special care to point out any differences between him and his predecessor and will be particularly interested to see if the medieval philosopher is any more complete in the areas just mentioned in which the Greek philosopher is lacking.

PART II

THE OBJECT AND NATURE OF MATHEMATICAL SCIENCE

IN ST. THOMAS AQUINAS

AND

A COMPARISON

OF HIS TEACHINGS WITH ARISTOTLE'S

CHAPTER I

THE OBJECT OF MATHEMATICS ACCORDING TO THOMAS AQUINAS

Agreeing with his Greek predecessor, St. Thomas Aquinas asserts that the objects of mathematics are quantities, or more precisely quantitative beings.

> [T]he mathematical sciences study one kind of being, quantitative being . . . .[1]

To speak of the object of mathematics as quantitative being is more accurate than to simply say quantity for, as we will discuss in the next chapter, Aquinas, like Aristotle, insists that quantity itself is not a substance but an accident, i.e. that which exists only in substance, and so it must always be understood in terms of substance. Though the object of mathematics, precisely speaking, is quantified being, the mathematician does not treat quantified being in terms of being but in terms of quantity and deduces the properties and attributes following upon it as quantitative.[2] Thomas states,

> The mathematical sciences rather consider those attributes which belong to this or that quantity . . . .[3]

As for the nature of quantity in general, Thomas describes it (It cannot be defined since it is an ultimate genus.) as consisting "in a certain multiplication of parts,"[4] and he also speaks of it as the order or arrangement of the parts

of substance.[5] We might note in passing that these descriptions of quantity are in terms of actuality: quantity is the actual multiplicity, order, arrangement of parts, which allows for a corresponding potency, viz. the divisibility of that which is quantitative. On the other hand, Aristotle's main description of quantity, found in his Metaphysics, is given in terms of its potentiality. Thomas repeats this description in his commentary.

> Quantity means what is divisible into constituent parts.[6]

Recall, however, that our analysis of this description in Part I, Chapter I (pages 9 and 10), concluded that for Aristotle the actual feature of quantity possessed by a body which allows for its divisibility is its actual unity of distinguishable parts, parts "outside of" parts, at least in thought. Thus, though the Stagirite emphasizes the potential aspect of quantity in his classic text, there is really no difference between him and St. Thomas on the general nature of this accident.

Aquinas also follows Aristotle in distinguishing the category of quantity from the closely related category of quality. (I say closely related since both of these categories refer to accidents which are absolutely intrinsic to and follow first upon substance; none of the other categories of accidents are such.)[7] Quantity is not that which accounts for things differing in kind (This is called analogously a quality.), nor is quantity a sensible quality, nor a habit or disposition, nor any of the other kinds of quality.[8] Quantity is simply that which, as the above texts assert, is a

multiplication, an arrangement, an ordering of parts.

To more fully understand the description of quantity as an arranged multiplication of parts, it will be helpful to determine more precisely the nature of the parts of which quantity is composed. In his commentary on Aristotle's description of the quantitative (See Aristotle's text on page 3, Chapter I, Part I.), Thomas, as we said, repeats with approval the Stagirite's statement that quantity is that which is divisible into parts which are in it (insunt); of which parts, Thomas says, either all or some are "unum aliquid hoc est aliquid demonstratum."[9] The phrase "aliquid demonstratum" is used by St. Thomas in other places to signify an individual substance and to mean the same as hoc aliquid or hoc signatum, which also mean an individual substance.[10] However, Thomas certainly can not mean that the parts of a quantity must be substances. For in this same text of his commentary, Aquinas cites a number as an example of a quantity — and in other places he refers to other mathematical entities such as circles, triangles, cubes, and so forth. Now none of these quantities are substances as we have said (and as we shall develop in Chapter II of this Part); they are attributes of substances. Neither can their constitutive parts, therefore, be substances. From the context of his remarks, it would seem that his phrase "aliquid demonstratum" is to be taken as synonymous with "unum aliquid." If so, then the point would be the same as that made by Aristotle in his description of the quantitative (Cf. Part I, Chapter I, pp. 3-5.), viz. that the parts into which

the quantitative is divisible must in some way be units, though they need not be individual substances, for one is convertible with being and so is found in every category of being. Whether this is the correct interpretation of Thomas's commentary, or whether it may simply be that he was faced with an inaccurate translation of Aristotle's text,[11] it is certain from many other passages that the parts which make up the quantitative and into which it is divided while they are units (else they would not be at all) need not be substantial units. The parts of a number, and indeed of any mathematical quantity, are not substances, for the quantity itself is not substance — as we have said.

Though the parts of a quantitative thing need not be substantial units they must be quantitative units else the unity of which they are the parts will not be, strictly speaking, quantitative. For example, a plurality made up of ten colors has as its unitative parts individual colors which, of course, are qualities. This plurality would not be quantitative in the strictest sense but only <u>per accidens</u> inasmuch as a color is a <u>per accidens</u> quantity.[12] (We will discuss this type of quantity later.) Likewise the three Persons in God do not make up something quantitative for each is not a quantitative unit.[13]

Following the Stagirite, St. Thomas asserts that quantity, which is divisible inasmuch as it consists of a multitude of parts, comes in two specifically different kinds: "[E]very divisible thing is either a number or a magnitude."[14] As Aristotle before him had said, magnitude is that type of

quantity which refers to the extension, the dimension, the actual "spreadoutness" (and corresponding divisibility) of a physical thing. Lacking this type of quantity, Thomas says, and he refers to Aristotle, a substance would be indivisible: "With the quantity gone all substance is indivisible."[15] In addition to magnitude, or extension, which is also termed by St. Thomas (following Aristotle) continuous quantity inasmuch as its parts have common boundaries, there is the kind of quantity called discrete whose parts do not have common boundaries.[16] This type of quantity is, of course, plurality expressed in number and is divisible into the units which make up the plurality. Now to repeat the point made earlier, these two kinds of quantity, which are unities of different kinds of parts, must be made up of parts which are themselves quantitative. We first made this point when discussing Aristotle's notion of quantity (Part I, Chapter I, page 6). It would be well, I think, to repeat our discussion here, particularly since Thomas's treatment of this material is much more thorough than the Stagirite's.

Let us discuss multitude, or number, first. St. Thomas holds that only those pluralities (and numbers that signify them) which result from the division of material things having continuous quantity are of the genus quantity. He agrees with Aristotle in affirming that the number of the genus quantity arises from division of material things with continuous quantity.[17] That multitude which simply results from different forms (e.g. a plurality of angels) "does not belong to a genus, but is transcendental in the sense in

which being is divided by one and by many."[18] Obviously, the units which make up this type of multitude as its parts are not quantitative but, as Aquinas says, are simply "opposite and diverse forms."[19] This latter type of multitude, called transcendental multitude by him, is what is referred to when one speaks of three persons in God, or twelve choirs of angels; it is, Thomas says, the only kind of multitude found among immaterial beings, beings which possess no real quantity.[20] However, transcendental multitude is not limited simply to immaterial beings. Or, to say the same thing another way, not every multitude which has reference to material beings and their attributes is necessarily of the genus quantity.

Every multitude, St. Thomas says, is a plurality of units. Insofar as the units which make up the multitude differ so will the multitudes which are the unities of these units differ.[21] He points out both in the Sentences and in Summa that the multitude (signified by number) which is of the category quantity must be composed of units which are not simply convertible with being (as is the transcendental unity) but which denote "an accident added to being."[22] He distinguishes from this quantitative multitude the multitude and number which is not limited to the category quantity but is transcendental inasmuch as it is a unity of units which are convertible with being, units which add to being only the negation of division. Now this latter type of multitude (which he naturally refers to as transcendental multitude and transcendental number since it is made up of transcendental

unity) can, therefore, be predicated of beings having no quantity in the categorical sense (called dimensive quantity), even of God Himself and the angels. Inasmuch as spiritual entities are units of some type, they can be numbered as though they contained quantity. Furthermore, this notion of transcendental multitude applies not only to spiritual substances but to any multitude which transcends, i.e. is not limited to, the genus quantity, for as St. Thomas says, "multitudo et unum circumeant omnia genera."[23] Transcendental multitude would apparently refer to any multitude made up of non-quantitative units. So, for example, the numerical terms used in reference to ten colors (category quality), five relations, six actions, seven virtues, since in each case they are said of a plurality of non-quantitative entities (qualities, relations, actions, in the examples given), would be said to be, strictly speaking, transcendental numbers rather than categorical.[24] (I say "strictly speaking" for it may be that these accidents could be considered per accidens quantity inasmuch as they could be present in a body having per se quantity — as ten colors, for example, could be present in ten different quantitative bodies or spread out in one body divisible nine times. Nevertheless, the colors would still be per se qualities and hence their plurality, their ten-ness, made up of qualitative units, would not be per se a quantitative plurality. The same would hold for any other accident which could be a per accidens quantity.[25] We will say more about this later.) Thus, we can conclude that every discrete quantity is a plur-

ality, a multitude, and can be numbered, but not every plurality, multitude, and number is a discrete quantity according to St. Thomas. Unless the units, the parts, of which the multitude is composed are per se quantitative, the multitude itself which is a unity of these parts will not be per se of the genus quantity. (Incidentally, the type of non-categorical quantity which he terms transcendental multitude and transcendental number is classified by him as one of the two kinds of virtual quantity — the kind analogous to the species of categorical quantity called discrete.)[26]

In reference to the species of categorical quantity termed continuous we can say that for St. Thomas every continuous quantity is a magnitude but not every magnitude is a continuous quantity. He speaks, for example, of spiritual magnitude, the magnitude of power and of act.[27] To illustrate, God is called infinite, an angel's intellect is said to be more powerful than a man's, a certain degree of heat is said to be greater than another.[28] This type of magnitude Thomas also classifies as a kind of virtual quantity; it is that species of virtual quantity analogous to the species of categorical quantity called continuous.[29] It is to this type of quantity, he says, that we refer when we speak in quantitative terms of the perfection of some nature or form in the order of being, e.g. when we say an angel is a greater being, more of a being, than a man or a rock, or when we speak of something being more or less hot or white or healthy, or when we speak of the increase of spiritual entities such as charity, or when we speak of the three Di-

vine Persons as equal to each other.[30] Even though more, less, increase, equality are properly quantitative terms,[31] they are used in the above examples to designate nonquantitative entities. What they are predicated of are not of the genus quantity but are substances or some other accident, or even beings transcending all categories in the case of God. And the "magnitude" in each of these cases is not a unity composed of continuous quantitative parts, in fact is not really composed of parts at all, but refers, Thomas says, fundamentally to the perfection or intensity of act, to the "amount" of being, whether accidental or substantial, possessed.[32] In distinction to this type of magnitude is that actually of the genus quantity, e.g. a line, a surface, a solid, whose parts are themselves quantitative and share a common boundary. Only this type of magnitude, a unity of parts which are continuous quantities, is per se of the category quantity.

The purpose of this analysis of Thomas Aquinas's notion of virtual or transcendental quantity has been to set forth more clearly his two basic kinds of categorical quantity, plurality and magnitude. To go any deeper into virtual quantity would take us too far from our topic.[33] This point must be stressed, however; it is categorical, not virtual quantity, which, according to Thomas, is studied by the mathematician. This is because strictly speaking virtual quantity is not of the genus quantity. Aquinas is very clear on this latter point and the reason for it.

Virtualis quantitas non est ex genere suo quantitas,

> quia non dividitur divisione essentiae suae; sed magnitudo ejus attenditur ad aliquid divisibile extra . . . .[34]

The quantity called virtual is not of its own essence (elsewhere he says secundum se)[35] divisible, but is divided according to something extrinsic to it which is divisible — as virtues are divisible and hence quantitative by reason of the division of their acts and objects,[36] and color is divisible and hence quantitative because of the surface in which it is. And so virtual quantities are not really of the genus quantity for "ratio quantitatis invenitur proprie in illis quae secundum se dividuntur."[37] Actually those quantities termed virtual are, as we have said above, properly (secundum se) of some other category, quality, substance, relation, etc.[38] Now since Aquinas accepts the Stagirite's notion of mathematical abstraction, viz. that the mathematician mentally leaves aside all the features of a physical thing which are qualities, actions, passions, etc., and concentrates only on its quantitative features, those features said to be quantities virtually would be left aside as a result of the abstraction. Furthermore, virtual quantities do not necessarily involve matter for, as we have seen him assert, they can be said of God, of charity, in other words, of nonmaterial entities. But the quantities studied in mathematics, both discrete and continuous, always involve matter, according to St. Thomas, both in being and in thought,[39] though the matter is different in each case. (Both the matter of mathematicals and mathematical abstraction will be discussed in detail in Chapter II of this Part.)

Therefore, the same conclusion follows: virtual quantities are not treated in the mathematical sciences which consider abstract quantities; only quantities per se of the genus quantity are considered.[40]

We mentioned earlier the basic agreement between St. Thomas and Aristotle on the notion of quantity (pages 173 ff.). We might ask, however, if the Stagirite distinguishes categorical from virtual quantity as thoroughly as does Aquinas. It is certainly true that the distinction between these two types of quantity is not as developed by Aristotle; however, he is quite aware of it as we showed in Chapter I, Part I. He distinguishes clearly between the unity of the category of quantity and the unity corresponding to being.[41] He distinguishes further between a plurality which is a unity composed of quantitative units and a plurality of qualities, of actions, etc., where the units are not per se quantitative.[42] This shows clearly that he grasps the difference between categorical and transcendental number.

The Stagirite also refers to the intensity or perfection of acts[43] and to their ability to be measured. The fact that he says that measure, which pertains primarily to the category of quantity, can be extended to categories other than quantity insofar as they partake in some way of quantity (viz. under the aspect of their being more or less)[44] shows that he is aware of this second type of virtual quantity, the type akin to physical magnitude. It is to this that we refer when we use quantitative terms such as more, less, increase, equal, of nonquantitative entities.

This would seem to indicate too that he recognizes the difference between this type of magnitude present in entities of some other category and the magnitude of the category quantity. Finally, for Aristotle, too, it could not be virtual quantity, i.e. the quantity found analogously in other categories, that is treated in mathematics, for the mathematician abstracts from all nonquantitative features, as we discussed in detail in Part I, Chapter II.

However, we also pointed out in Part I (Chapter I, page 6) that the Stagirite seems to have never applied his distinction between quantitative and nonquantitative units and pluralities to the quantities studied in mathematics. In fact he often neglects to affirm that the units and numbers treated by arithmetic must be quantitative units and numbers. St. Thomas, on the other hand, is perfectly clear, as we have shown, that it is only quantitative ones and pluralities which mathematics considers. Thus, Aquinas does have more detail on and development of the distinctions enunciated by the Greek philosopher and he clearly applies them to mathematical quantities. Nevertheless, it is the Stagirite's principles and distinctions that he developes and applies. Indeed, Thomas himself views his doctrine on this matter as following that of the Stagirite's, for he explicitly refers to Aristotle's teaching on the perfection or intensity of act as being the same as his notion of virtual quantity in the sense of magnitude.[45] Also Aquinas's notion of measure and its applicability to entities other than quantitative is taken from the Stagirite.[46] We find,

then, that the two philosophers are in agreement on the object of mathematics — categorical quantity, and on the nature of this object. For both men, categorical quantity, either in the form of magnitude or of multitude, refers to an actual unity of many quantitative parts, parts which are separable in some way from the whole becoming by division quantitative units of some kind.

One characteristic of quantity mentioned by St. Thomas which is particularly important for mathematics, a characteristic never explicitly enunciated by Aristotle, is that of its ontological relation to other accidents. Aquinas points out in many places that:

> Among accidents, however, there is a certain order to be considered. For, among all the accidents, that inhering more closely to the substance is quantity which tends to measure. Then the qualities are received in the substance with the quantity as medium — color, for example, with the surface as medium . . . .[47]

and then come

> . . . actions and passions [and relations], and the motions consequent upon sensible qualities.[48]

Quantity, then, is ontologically prior to the other accidents.[49] This priority can be seen briefly in this way. All sensible qualities, in order to be sensible, must inhere in something spatially extended — hence must inhere in quantity. Furthermore, any corporeal activity is located in extended parts and has as its basis qualities that are already extended. The relations of bodies to each other depend upon their quantity, their quality, and/or actions. And the other accidental features which are not inherent in the thing

but arise from extrinsic denomination depend in turn upon the aforementioned inherent accidents.[50] Aquinas even likens quantity to matter inasmuch as it is the subject of other accidents for they inhere in and determine it just as form inheres in and determines matter.[51] Of course, to call the quantity of a thing matter is to mean matter only in an analogous sense. As we said earlier St. Thomas fully agrees with Aristotle that quantity is, like quality, an accident and so has being only by inhering in substance. The point is, however, though quality and the other accidents have being ultimately only by inhering in substance, they inhere through the mediation of quantity which is naturally prior to them.[52] This ontological priority of quantity, which as I have said is never explicitly mentioned by Aristotle (see above Part I, Chapter II, page 72), will be very important when we come to treat of mathematical abstraction in the next chapter.

We mentioned that Thomas likens quantity to matter inasmuch as it is the subject of other accidents. Actually there is even more to the relation between quantity and matter. As we pointed out when dealing with Aristotle's notion of quantity (Part I, Chapter I, page 14) magnitude, continuous quantity, can be present only in a material being, i.e. in a being having a principle of potency. Aquinas states, "[E]verything possessed of quantity is in a certain manner in potency."[53] Since the potentiality of physical things is rooted in matter, the principle of potency, magnitude demands matter.[54] Only material things can have the magnitude

of the category of quantity. In this connection Thomas often speaks of quantity following upon (consequens) matter rather than upon form (and in these texts he is clearly speaking of categorical quantity.)[55] I interpret this to mean that matter is ontologically prior to and demanded by quantity, inasmuch as quantity means potentiality in the form of divisibility. All qualities, on the other hand, do not necessarily require potentiality. Aquinas points out that even pure forms can have qualities. Of course the qualities of corporeal things, inasmuch as they inhere in quantity which in turn demands matter, could be said to demand matter. However, quality simply as such does not require matter, for some immaterial entities are qualified, and so St. Thomas will never say that quality "follows upon" matter.[56]

We have been talking of the necessary relation of continuous quantity and matter. What should be said about discrete quantity and matter? We pointed out above (page 181) that Thomas holds that both species of categorical quantity necessarily involve matter. We also noted earlier that Aquinas, following Aristotle, holds that discrete quantity results from the division of continuous quantity. For example, a stick which has parts which are continuous quantities can be actually divided into separate parts each of which is itself a discrete quantity, each of which itself has parts which are continuous. Now if discrete quantity results from a division of continuous quantity and continuous quantity demands matter, as we have shown above, then

it would follow that discrete quantity too ultimately demands matter. This is what St. Thomas asserts for he states that quantitative number is found only in things having continuous quantity, only in material things.

> All plurality is a consequence of division. Now division is twofold: one is material, and is division of the continuous; from this results number, which is a species of quantity. Number in this sense is found only in material things which have quantity.[57]

Thus, Aquinas' position is clear: categorical quantity, both discrete and continuous, demands matter. The necessary relation of quantity to matter will be developed at length in the following chapter.

In reference to Aristotle, we pointed out in Part I, Chapter I, page 15, that he does hold that discrete quantity arises from a division of continuous quantity. And we have discussed at length in Chapter II, Part I, how he does include matter, intelligible matter, in both discrete and continuous mathematical quantities, and does say that categorical quantity <u>really</u> (as distinct from mentally) exists only in material things. There is no question, then, but that the Stagirite does see a close relation between categorical quantity and matter. However, he never develops this relation nor does he state, to my knowledge, that quantity "follows upon matter."[58]

In concluding this first chapter dealing with Thomas Aquinas' notion of quantity, the object of mathematics, we should repeat again that he is in general agreement with Aristotle on the nature of this accident. Yet we must also acknowledge that Thomas, while sharing many of the Stagirite's

fundamental principles, has a much more developed doctrine on quantity than he does. This is true in regard to the distinction of categorical and virtual quantity (and in his affirmation that mathematics considers only the former), the ontological priority of quantity to other accidents, and the necessary relation of quantity to matter.

FOOTNOTES

[1]St. Thomas Aquinas, Commentary on the Metaphysics of Aristotle, translated by John P. Rowan in "Library of Living Catholic Thought," Vol. I (Chicago, 1961), IV, L. 1, 532. Unless otherwise noted, all further references to Thomas's Commentary on the Metaphysics will be to this edition.

[2]Maritain, The Degrees of Knowledge, pp. 143 ff., shows clearly the different ways the philosopher of nature, the metaphysician, and the mathematician study quantity. On this point, see also Maziarz, The Philosophy of Mathematics, p. 198.

[3]In XI Metaphysics, L. 4, 2206. See also 2202.

[4]Aquinas, Summa Contra Gentiles, I, 69, translated into English under the title On the Truth of the Catholic Faith by A. C. Pegis, J. P.Anderson, V. J. Bourke, and Charles O'Neil, 5 vols. (New York, 1954-1956). Unless otherwise noted, all further references to the Summa Contra Gentiles will be to this edition.

[5]Aquinas, Summa Theologica in Basic Writings of Saint Thomas Aquinas edited and annotated, with an introduction, by Anton C. Pegis, Vol. I (New York, 1945), I, 14, 12, ad 1. Unless otherwise noted, all further references to the Summa Theologica will be to this edition. Aquinas, Commentary on the De Trinitate of Boethius, Qq. V and VI, The Division and Methods of the Sciences, translated by Armand Maurer (Toronto, 1953), V, 3, ad 3. Unless otherwise noted, all further references to the De Trinitate will be to this edition. Aquinas, De Veritate under the English title Truth, translated from the definitive Leonine text by Robert W. Mulligan, S. J., 3 vols. (Chicago, 1952), II, 9, c. Unless otherwise noted, all further references to the De Veritate will be to this edition. See also Maziarz, The Philosophy of Mathematics, p. 193 and Bertrand Mahoney, The Notion of Quantity in a Thomistic Evaluation of the Sciences (River Forest, Ill., 1951), pp. 2, 19, 46.

[6]In V Metaphysics, L. 15, 977. For Aristotle's definition, cf. Metaphysics, V, 13, 1020a 8-9. Smith, St. Thomas on the Object of Geometry, p. 33 speaks of the two descriptions of quantity — one he calls formal (in terms of act), one material (in terms of potency).

[7]In V Metaphysics, L. 9, 891-2.

[8]For the kinds of quality according to St. Thomas, cf. Book V of his Commentary on the Metaphysics of Aristotle, L. 16 and Owens, . . . Christian Metaphysics, pp. 170-178.

[9]In V Metaphysics, L. 15, 977.

[10]Roy J. Defarrari, Sister M. Inviolata Barry, A Lexicon of St. Thomas Aquinas (Baltimore, Md., 1946), gives these meanings and synonyms for aliquid demonstratum, p. 49.

[11]It might be pertinent to note that the Latin translation of Aristotle's Metaphysics which Thomas used for his Commentary translates the Greek τοδε τι (usually translated in English as "a this") as hoc aliquid, which as we said meant for Thomas a substance. However, the Greek term τοδε τι ("a this") does not necessarily have this connotation as we pointed out in Chapter I, Part I. It may be, in other words, that Thomas simply had a faulty translation of Aristotle's text on his hands and stayed with it in his commentary. Or it may be, as I suggested in my text, that he uses the phrase aliquid demonstratum (equivalent to hoc aliquid) in an analogous sense, i.e., not meaning necessarily an individual substance but a unit of some type. Either interpretation seems possible.

[12]In V Metaphysics, L. 15, 984.

[13]S. T. I, 30, 3c; In I Sententiarum, D. XIX, Q. I, art. I, ad 1 (Parma, VI, p. 160) and D. XVII, Q. II, art. I, c (Parma, VI, p. 143).

[14]Aquinas, Commentary on Aristotle's Physics, translated by Richard J. Blackwell, Richard J. Spath and W. Edmund Thirlkel (New Haven, 1963), III, L. 7, 345. Unless otherwise noted, all further references to the Commentary on Aristotle's Physics will be to this edition.

[15]C. G., IV, 65, 4. See also S. T., I, 50, 2c.

[16]In V Metaphysics, L. 15, 978.

[17]In I Sententiarum, D. XXIV, Q. I, art. III, c (Parma, VI, p. 203). In III Physics, L. 12, 393-6; S. T., I, 30, 3c.

[18]S. T., I, 30, 3c; on this same point see also Quaestiones Disputatae De Potentia, Q. IX, art. VII, c (Parma, VIII, p. 191); In I Sententiarum, D. XXIV, Q. I, art. III, c (Parma, VI, p. 203). Cf. Mahoney, The Notion of Quantity . . ., pp. 17-18 and Maziarz, The Philosophy of Mathematics, p. 195.

[19]Ibid.

[20]Ibid.

[21]Quaestiones Disputatae De Potentia, Q. IX, art. VII, c (Parma, VIII, p. 191).

[22]S. T., I, 30, 3c. See the whole of this article.

". . .[A]ll plurality is a consequence of some division. Now division is twofold. One is material, and is division of the continuous; from this results number, which is a species of quantity. Number in this sense is found only in material things, which have quantity. The other kind of division is called formal, and is effected by opposite or diverse forms; and this kind of division results in a multitude, which does not belong to a genus, but is transcendental in the sense in which being is divided by one and by many. Only this kind of multitude is found in immaterial things.

. . . . . .

". . .[M]ultitude in a transcendental sense . . . has relation to the many of which it is predicated, as one convertible with being is related to being; which kind of oneness does not add anything to being, except only a negation of division, ... For one signifies undivided being. . . . In the same way, when we speak of many things, multitude in this latter sense points to those things as being each undivided in itself. But number, if taken as a species of quantity, denotes an accident added to being; as also does one which is the principle of that number."

In I Sententiarum, D. XXIV, Q. I, art. III, c (Parma, VI, p. 203); Quaestiones Disputatae De Potentia, Q. IX, art. VII, sed contra 5 (Parma, VIII, p. 190) and corpus (Parma, VIII, p. 191). Also see Alvarez's discussion of this point, "La filosofia della matematica . . .," 144.

[23]Quaestiones Disputatae De Potentia, Q. IX, art. VII, sed contra 5 (Parma, VIII, p. 190).

"[O]mnis divisio vel distinctio aliquam multitudinem causat. Non autem omnis multitudo est numerus qui est species quantitatis; cum multitudo et unum circumeant omnia genera. Non ergo omnis divisio vel distinctio sufficit ad constituendum numerum qui est species quantitatis, sed sola illa divisio quae est secundum quantitatem, qualis non est distinctio relationum."

[24]In Ibid., Thomas affirms that a multitude of qualities and of relations does not make up numbers which are of the genus quantity. Also In I Sententiarum, D. XVII, Q. II, art. I, c and ad 1 (Parma, VI, p. 143) he speaks of virtual quantities, multiplicities, which are not "ex genere suo"

quantities (Cf. texts on pages 180 and 181.), like a plurality of virtues, of white colors, of heat.

[25]Ibid. See also In V Metaphysics, L. 15, 983.

[26]In I Sententiarum, D. XVII, Q. II, art. I, ad 2 (Parma, VI, p. 143), text given in footnote 29. See also Owens, . . . Christian Metaphysics, p. 168 and James Anderson, The Bond of Being (St. Louis, 1949), pp. 295-6, both of whom classify transcendental number under virtual quantity.

[27]C. G., I, 43, 1 and 2.

[28]S. T., I, 42, 1, ad 1.

[29]In I Sententiarum, D. XVII, Q. II, art. I, ad 2 (Parma, VI, p. 143). Occasionally, St. Thomas speaks as if the phrase virtual quantity is simply equivalent to this type of magnitude and does not include transcendental multitude. For example, see De Veritate, 29, 3c; S. T., I, 42, 1, ad 1. However, in the Sentences in the place just cited, he does indicate that there are two species of virtual quantity analogous to the two species of categorical quantity. This is his text:

"[Q]uantitas virtutis attenditur dupliciter: vel quantum ad numerum objectorum, et hoc est per modum quantitatis discretae; vel quantum ad intensionem actus super idem objectum, et hoc est sicut quantitas continua . . . ."

Actually, whatever the terms should refer to, and Thomas is not perfectly consistent in his usage, his doctrine is clear. There are two kinds of noncategorical quantity — one which he calls transcendental multitude, one which he calls magnitude.

[30]All of these expressions are found in one or the other of the following places: De Veritate, 29, 3c; S. T., I, 42, 1, ad 1; C. G., I, 43, 2; S. T., I-II, 52, 1c, literally translated by the Fathers of the English Dominican Province (3rd ed.; London, 1938); In I Sententiarum, D. XVII, Q. II, art. I, ad 1 (Parma, VI, p. 143). See also Anderson, The Bond of Being, pp. 295-296.

[31]S. T., I-II, 52, 1c (Dominican trans.).

[32]S. T., I, 42, 1, ad 1.

"[Q]uantity of **virtue** . . . is measured according to the perfection of some nature or form: to this sort of quantity we allude when we speak of something as being more, or less, hot, inasmuch as it is more, or less, perfect in heat. Now this virtual quantity is measured firstly by its

source — that is, by the very perfection of the form or nature: such is the greatness of spiritual things; just as we speak of great heat because of its intensity and perfection."

On this same point, refer to S. T., I-II, 52, 1c (Dominican trans.); De Veritate, 29, 3c; Mark Heath, "Can Qualities Be Measured?" 34 and 51; Maziarz, The Philosophy of Mathematics, p. 195.

33Two questions in regards to virtual quantity which come to mind are: (1) What is the exact relation of per accidens to virtual quantity? There is definitely a distinction between the two for St. Thomas (Cf. In I Sententiarum, D. XVII, Q. II, art. I and ad 1 (Parma, VI, p. 143); (2) Is virtual quantity in the sense of magnitude a transcendental property of being, since every being is being in some degree or "amount"? Can it be reduced to one of the more commonly known transcendentals?

34In I Sententiarum, D. XVII, Q. II, art. I, c (Parma, VI, p. 143).

35In I Sententiarum, D. XIX, Q. I, art. I, ad 1 (Parma, VI, p. 143).

36Ibid.

"[R]atio quantitatis invenitur proprie in illis quae secundum se dividuntur; et sic Deo non convenit. Invenitur etiam quodammodo in illis quorum divisio attenditur secundum ea quae extrinsecus sunt, sicut virtus dicitur divisibilis et quantitatis rationem habens ex ratione et divisione actuum et objectorum."

37Ibid.

38Quaestiones Disputatae De Potentia, Q. IX, art. VII, sed contra 5 (Parma, VIII, p. 190); In I Sententiarum, D. XVII, Q. II, art. I, c (Parma, VI, p. 143).

"Virtualis quantitas non est ex genere suo quantitas, quia non dividitur divisione essentiae suae; . . . Sed ex genere suo est forma accidentalis in genere qualitatis, vel forma substantialis; . . . ."

39Quaestiones Disputatae De Potentia, Q. IX, art. VII, c (Parma, VIII, p. 191); S. T., I, 30, 3c; S. T., I, 11, 3, ad 2; S. T., I, 85, 1, ad 2. For a complete discussion of the presence of matter in the objects of mathematics, see below Chapter II, the section entitled "The Constitution of the Essence of the Objects of Mathematics — Intelligible Matter."

[40]We shall see in Chapter IV, however, that virtual quantities can be studied in the intermediate mathematical sciences.

[41]Metaphysics, X, 2, 1053b 24 ff. See also the references in the following footnote.

[42]See Metaphysics, X, Chapter 1, where he discusses measure and says that the basic unit of measure, and hence the number which is a plurality of units, is different in each category.

[43]Physics, VII, 3, 246a 13-15.

[44]Metaphysics, X, 1, 1052b 18-31. See Mark Heath's discussion of this, "Can Qualities Be Measured?" 34 ff.

[45]De Veritate, 29, 3c.

"Now there are two kinds of quantity: dimensive, which is referred to extension; and virtual, which is referred to intensity; for the excellence (virtus) of a thing, is its perfection, as the Philosopher teaches. 'Anything is perfect when it attains its proper excellence'; and the virtual quantity of each form is considered according to the degree of its perfection."

[46]In X Metaphysics, L. 2, 1938-1942.

[47]C. G., IV, 63, 9.

[48]In II Physics, L. 3, 161.

[49]Much of Thomas's discussion of this ontological relation of the accidents, incidentally, is found in his treatments of the Eucharist where he discusses the subject of the sensible qualities of bread which persist after transubstantiation. He concludes that the subject of these qualities is quantity. C. G., IV, 63, 9; S. T., III, 77, 2c (Dominican trans.). For a discussion of this point, see Maritain, The Degrees of Knowledge, pp. 142 ff. and Maziarz, The Philosophy of Mathematics, p. 198.

[50]This analysis is from Owens, . . . Christian Metaphysics, pp. 168-9 and Chapter 14. For a more complete treatment of this point, see his discussion of each of the categories, Chapters 12-14. Chapter 14 shows clearly how each of the categories arising from extrinsic denomination depends on those arising from intrinsic denomination — p os ture depends on where which in turn depends on quantity; when depends on movement which depends on quality; habitus on quality, quantity, action and/or relation.

[51]In IV Sententiarum, D. XII, Q. I, art. I, Quaestiuncula III, Solutio III, c (Parma, VII, p. 655).

"[P]rima accidentia consequentia substantiam sunt quantitas et qualitas; et haec duo proportionantur duobus principiis essentialibus substantiae, scilicet formae et materiae . . . sed qualitas ex parte formae. Et quia materia est subjectum primum quod non est in alio, forma autem est in alio, scilicet materia; ideo magis appropinquat ad hoc quod est non esse in alio, quantitas quam qualitas, et per consequens quam alia accidentia."

See also Geiger, "Abstraction et separation . . .," 35.

[52]In IV Sententiarum, D. XII, Q. I, art. I, Quaestiuncula III, Solutio III, c (Parma, VII, p. 655); S. T., III, 77, 2c (Dominican trans.); C. G., IV, 63, 9.

[53]C. G., I,20, 3.

[54]Aquinas, The Principles of Nature in An Introduction to the Philosophy of Nature, compiled by R. A. Kocourek (St. Paul, Minn., 1948), #2-5. All further references to The Principles of Nature will be to this edition unless otherwise noted. De Trinitate, V, 4c.

[55]In III Physics, L. 5, 322; In V Metaphysics, L. 9, 892:

"A predicate is referred to a subject in a second way when the predicate is taken as being in the subject, and this predicate is in the subject either essentially and absolutely and as something flowing from its matter, and then it is quantity; or as something flowing from its form, and then it is quality; . . ."

In IV Sententiarum, D. XII, Q. I, art. I, Quaestiuncula III, Solutio III, c (Parma, VII, p. 655) (text in footnote 51). On this point, cf. Smith, St. Thomas on the Object of Geometry, p. 35.

[56]Quaestiones Disputatae De Potentia, Q. IX, art. VII, c (Parma, VIII, p. 191).

"Unde patet quod nulla species quantitatis potest in rebus spiritualibus convenire, nisi secundum metaphoram. Qualitas autem sequitur formam; unde quaedam qualitatis sunt omnino immaterialis, quae attribui possunt rebus spiritualibus."

Actually the expression "quantity follows upon matter" (consequens materia) strikes me as very ambiguous. It could be taken in a very loose sense, viz. simply that quantity is akin to matter. Just as matter is the subject of and is determined by form, so quantity is the subject of and is determined by quality. This seems to be one meaning

Thomas has in mind in the Sentences. (In IV Sententiarum, D. XII, Q. I, art. I, Quaestiuncula III, Solutio III, c Parma, VII, p. 655) (Text given in footnote 51)

A stricter meaning for "quantity following upon matter" would be that given in my text, viz. that quantity is ontologically posterior to and presupposes matter, the principle of potency; quality, on the other hand, does not necessarily presuppose matter, though corporeal qualities do.

"Quantity following upon matter" might also refer to the fact that quantified matter is the basis of the individuation of forms according to Aquinas (Cf. De Trinitate, V, 3, ad 3; V, 2, ad 1; C. G., IV, 65, 4; In VII Metaphysics, L. 2, 1283), and so quantity in this sense would be analytically, though not really, prior to the form it individuates and hence must follow upon matter rather than form. (In II Sententiarum, D. III, Q. I, art. 4c Parma, VI, p. 415) (For a very complete discussion of the role of quantity in individuation and its relation to matter and form, see the doctoral dissertation of Joseph Bobik, Saint Thomas on the Individuation of Bodily Substances Notre Dame, 1953 , Chapter II.)

Again, the statement could mean that quantity is in some way really derived from matter. However, I would interpret this derivation only in the sense given in my text, viz. quantity presupposes matter as principle of potency. Certainly, quantity could not be derived from matter as a property is from form since matter in itself is purely potential and can only function as a passive principle. (C. G., II, 73, 81, 83. See also Robert A. O'Donnell, "Individuation: An Example of the Development in the Thought of St. Thomas Aquinas," New Scholasticism, XXXIII 1959 , 63.)

Actually, in the last analysis, all accidents according to St. Thomas are a consequence of, follow upon, form (S. T., I, 7, 3c; In I Physics, L. 9, 66) and none of them are actually present in matter before form. (S. T., I, 76, 6c) All come to matter through form though the first which can be analytically distinguished following upon form is quantity. In this sense, of all the accidents it is closest to matter. (In V Metaphysics, L. 15, 983; S. T., I, 76, 6 ad 1 and 2; In I Sententiarum, D. VIII, Q. V, art. II [Parma, VI, pp. 79-80]) (It is this analytical moment, incidentally, that Bobik sees as the quantitative matter which is the principle of individuation for St. Thomas. Bobik states,

"This analytical moment in which the composite is so considered, i.e. as conferring quantity in act by its substantial form, and by its matter, at once irreceivable and receptive, receiving both the substantial form and the quantity so conferred, and this prior to the invasion of

any qualitative determination including figure and measure, is the moment in which St. Thomas sees the principle of individuation." (op. cit., pp. 261-262)

See his whole discussion, Chapter II, especially, pp. 190-207. O'Donnell's article on this point is also interesting. He too roots individuation in form; see especially his summary, 63-67. See also Gabriel, "Matière intelligible . . .," 191 and footnote thereon.)

Furthermore, even though all accidents are a consequence of substantial form, form just of itself does not have attributes of quantity. Pure forms are nonquantitative. Only forms in matter, form-matter composites, can be quantified. On the other hand, form of itself can have attributes of quality; it need not be in matter. Therefore, quantity follows upon form as a consequence of its being in matter, not so for quality. In this sense then, quantity, though like all accidents following upon form, could be said to be a consequence of matter.

[57] S. T., I, 30, 3c. He makes the same point in In I Sententiarum, D. XXIV, Q. I, art. III, c (Parma, VI, p. 203).

[58] One of the reasons that Aristotle does not develop the relation of quantity to matter may be that he never discussed how the categories are derived. It is in his treatments of the derivation of the ten categories that St. Thomas mentions quantity as following upon matter. (See texts cited in footnote 55.)

CHAPTER II

THE PRECISE NATURE OF THE QUANTITIES WHICH ARE THE OBJECTS OF MATHEMATICS

Distinction of Mathematical Quantities from Real Quantities

Agreeing with the Stagirite, Thomas Aquinas holds that the quantities considered in mathematics are gained by abstraction. He asserts this doctrine in many places; one text will be sufficient to clearly show his fundamental acceptance of this notion originated by Aristotle.

> [T]he science of mathematics studies "those things which are gotten by taking something away," i.e. abstract things . . . . For the science of mathematics carries on its investigations by removing from the scope of its study all sensible qualities, such as lightness, heaviness, hardness, softness, heat and cold, and all other sensible qualities, and retains as its object of study only the quantified and the continuous. . . .[1]

St. Thomas follows Aristotle in affirming that mathematical quantities are not to be identified with quantified natural things for the former lack all sensible qualities (as the above text states); they are in no real place;[2] they are not always three-dimensional (e.g. lines, surfaces, points, numbers) as physical things are; they are immobile (i.e. unable to undergo accidental change of quantity, quality, and place); and they are free from sensible matter.[3]

> Sensible matter is what pertains to the sensible qualities, hot and cold, rare and dense and the

> like; and with this matter natural bodies are concreted. Now the objects of mathematics abstract from this kind of matter.[4]

Concerning the immobility of mathematicals we should note that we mean this word in its most precise sense. It is true of St. Thomas as it was of Aristotle — mathematicals are asserted to be immobile, not unchangeable.[5]

Now according to Aquinas, to be immobile in the strictest sense, as Aristotle has shown, means to be free from accidental changes of quality, quantity, and place; it does not necessarily mean to be free from non-accidental changes.[6] In the less strict sense, Thomas, like Aristotle before him, uses immobile to mean free from all change, substantial as well as accidental; hence, he refers to God as immobile, meaning completely unchangeable.[7] It seems clear that when he refers to mathematicals as immobile, he means it in the strictest sense for there are many texts in which he refers to some mathematical entities as coming to be from other mathematical entities and correspondingly some being destroyed to form new mathematical entities, in other words as undergoing what for them is analogous to substantial change. For example, he speaks of one number arising from many different numbers by addition, multiplication, subtraction, and division,[8] and also of the division, multiplication, addition, and subtraction of mathematical magnitudes to generate other magnitudes,[9] e.g. the division of a circle into its segments, semicircles and quarter circles, the multiplication of a line by itself to result in a square, the addition of one line to another to make a larger.

But to cause by division or combination a new number or figure to come to be from a prior number or figure which in turn ceases to be is analogous to the substantial change of physical things whereby a given being or beings ceases to be and from it comes a new being.[10] (I say analogous since mathematical quantities are not as such extramental physical substances as we shall discuss below and so do not really undergo change since they do not really exist extramentally.)[11]

Furthermore, as was pointed out in the previous chapter (and in Part I, Chapter II, pp. 29 and 30), the very definition of quantity as given by Aristotle and accepted by St. Thomas, viz. that which is divisible into parts, affirms that a given mathematical quantity can be divided up into other quantities which before its division were present in it as its parts. In like manner, this definition would imply that a mathematical quantity could be formed (generated) by combining other quantities which would come to be its parts and in the process would lose their own individual identity. (Two 3's make 6, but 6 is not two 3's but one 6, a statement of Aristotle's which St. Thomas repeats.)[12] All of these considerations indicate that for Aquinas, as for Aristotle, mathematical quantities can undergo "substantial" change, amd thus that, when he calls them immobile, he means it in the strictest sense of the term.[13]

As we noted on pages 198 and 199 of this chapter, Thomas, like Aristotle, holds that the objects of mathematics are free from sensible matter, viz. matter considered as the

substrate of sensible qualities present in natural things.[14] But as immobile, mathematical quantities, of course, are even more free from the matter present in physical things, the matter viewed now not just as the substrate of qualities but as their principle of motion.[15] In fact, it is precisely on the basis of their separability in thought, i.e. by abstraction, from matter and motion that St. Thomas frequently distinguishes mathematical quantities from quantified physical things, which are neither really nor in thought separable from sensible matter and principles of motion.[16] Incidentally, though we say that mathematical quantities are free from matter we must keep in mind that Thomas will follow the Stagirite in asserting that mathematicals do contain a kind of matter, viz. the matter which Aristotle called intelligible. (Indeed, Thomas, like Aristotle, refers to the parts of mathematical figures, the segments of a line, lines of a figure, acute angles of a right angle, and the parts of numbers, units or lesser numbers, as their <u>matter</u>.[17] We will say more on this mathematical matter later.)

Mathematics Studies Abstracted Quantity — i.e., <u>Real</u> Quantity but Not <u>Qua Real</u>

We pointed out in the beginning of this chapter that Aquinas follows Aristotle in holding that mathematical quantities are gained by abstraction from physical things. The only quantities which really exist are those which are accidental features of existing physical things. St. Thomas repeats again and again that quantities, the objects of mathe-

matics, are not according to their real existence separate from physical matter and motion.

> According to their being [*esse*] they are not able to be separated from sensible matter.[18]

and again

> But those things which mathematics considers are not separable from matter and motion in being . . . .[19]

There is no difference between Aquinas and Aristotle on this point: the only really existing quantities are attributes of physical things with matter and motion. Furthermore, they both agree that the mathematician by his abstraction separates in his thought the quantities of things from the sensible matter and motion in which they actually exist. (See the first text of this chapter.) Thomas affirms that mathematicians,

> . . . though they consider forms that exist in sense-perceptible matter, do not include such matter in their definitions. The forms referred to are such as curved, straight, and so on, which, though they exist in matter and are, in fact, inseparable from it, are not, by the mathematician, regarded under their sense-perceptible aspect.[20]

He puts it this way in commenting on Aristotle's contrast between the way the philosopher of nature considers physical magnitudes and the way the mathematician does.

> [T]he mathematician and the natural philosopher treat the same things, i.e. points, and lines, and surfaces, and things of this sort, but not in the same way. For the mathematician does not treat these things insofar as each of them is a boundary of a natural body, nor does he consider those things which belong to them insofar as they are the boundaries of a natural body. But this is the way in which natural science treats them . . . . Because the mathematician does not consider lines and points, and surfaces, and things of this sort, [quantities] and their accidents, insofar as they are the boundaries of a natural body, he is said to abstract from sensible and natural matter.[21]

Again, in speaking of the way the mathematician considers a physical bodily substance, he says,

> [T]he mathematician does not consider the body which is found in the category of substance, according as its parts are matter and form, but he rather considers body according as it is in the category of quantity constituted by three dimensions; and body in this sense is related to body in the category of substance . . . as an accident to its subject.[22]

Clearly Aquinas and Aristotle are in agreement on this point. The mathematician considers quantities apart from the physical things in which they really exist. He treats physical things in terms only of their magnitude and quantitative plurality separating in thought these accidental features from the physical things even though magnitude and plurality do not really exist separate from physical things.

The quantitative aspects of physical things which the mathematician mentally separates are actually for St. Thomas, as they were for Aristotle, accidental features which are grasped by more than one sense, viz. common sensible features.[23] This is not to say that Aquinas simply identifies mathematical quantities with common sensible features, no more than the Stagirite did (Part I, Chapter II, pp. 59 and 60).[24] Though the quantities studied by mathematics are abstracted real quantities, precisely because they are abstracted by the mathematician they lack features which they possess as really existing. Abstracted mathematical quantities and physical quantities (which are common sensibles) can not be identified for many reasons. As we have seen, mathematical quantities lack sensible qualities, physical quantities do not; indeed, common sensibles are grasped by

the senses only through the properly sensible qualities which inhere in them.[25] Moreover, physical quantities (common sensibles) are in motion and place inasmuch as they are features of bodies in motion and place; mathematical quantities are not. Physical quantities (common sensibles) are three-dimensional; not all mathematical quantities are. Physical quantities (common sensibles) are in sensible matter; mathematical quantities are not but are to some extent immaterial. Thus, for example, the two-dimensional, immobile, qualityless rectangle studied in geometry is not identical to the three-dimensional brown, hard, cool, rectangular wood surface of the desk in front of me, though the abstraction which results in the former can be founded upon our sensible perception of the latter.

It follows from this that the quantities dealt with by mathematics are found with the peculiar characteristics which they possess only in the mind of a mathematician who has performed an act of abstraction.[26] The texts cited above, which refer to the mathematician separating physical quantities in thought from physical things, show very clearly that this is Aquinas's position. It is only in thought that quantities are nonsensible, immobile, nowhere, uni, bi, or even no dimensional, and to some extent immaterial. Indeed, in a passage of his Commentary on the Sentences, St. Thomas goes so far as to describe mathematicals in terms that in other places he uses to describe entia rationis.[27] (Incidentally, this passage was written by Aquinas late in his life and inserted in his Commentary. It should, there-

fore, give his mature position on the subject.)[28] This is not to say that he holds that mathematical quantities are simply created by man's intellect — for this would be contrary to his view that secundum esse quantities are in physical matter and that these quantities are the bases of the mathematician's abstraction. Rather, St. Thomas explains that like genus, species, etc., a mathematical is not simply a likeness of realities existing outside the soul but instead is a consequence of man's way of knowing some things outside the soul.

> [E]st aliquid quod consequitur ex modo intelligendi rem quae est extra animam.[29]

Things of this type, he says, are intentions which our intellect comes upon (adinvenit) because of its knowledge of extramental things. And he adds that the proximate foundation for such intentions is not "in re, sed in intellectu, tamen remotum fundamentum est res ipsa."[30] As Father Maurer points out in his analysis of this passage,[31] the expressions used here by Aquinas to describe mathematical entities are the same as those he uses in other places to describe entia rationis. In his Commentary on the Metaphysics for example,[32] St. Thomas states that in contrast to a natural being an ens rationis is strictly speaking an intention which reason comes upon (adinvenit) in the real thing it considers, an intention which is not found in the nature of things but is a consequence of the consideration of reason. Thus, it does not seem inaccurate to say, as Maurer does,[33] that mathematicals are beings of reason. What I mean by this is sim-

ply what I have already stated, viz. mathematical quantities as such, i.e. as immobile, nonsensible, etc., exist only in and by the mind itself. As Thomas says in the Scriptum text, their proximate foundation, that which immediately gives them their very being, is the mind itself. (This is not true of the beings studied in either physics or metaphysics. They exist in their own right apart from any act of a human intellect.) Without its act of abstraction, without its "liberation" so to speak of the quantities of things from their sensible, mobile, material existence, there would be no mathematical quantities. And yet he also says their remote foundation is real things.[34] As we have noted, it is the quantities of real things that the mathematician abstracts; indeed, there exist no other quantities for him to know.

Thus, like Aristotle, Thomas can say that the objects of mathematics are not separate from natural things but depend upon them secundum esse. Yet, at the same time, he can describe them as entia rationis having a proximate foundation in the intellect and say that "they have a separate existence only in the reason, inasmuch as they are abstracted from motion and matter."[35] The point is — though it is real quantities which the mathematician considers, he does not consider them qua real but qua mental, i.e. he treats them as separate from real things in terms of the attributes which they possess because they are mentally separated. Aquinas says, "The science of mathematics treats its object as abstracted by reason [secundum rationem]."[36]

It is true that the only quantities that really exist extramentally are accidents of physical things but it is not as such that the mathematician considers them, as we have said. They exist with their properly mathematical characteristics only in the mind of the man who has abstracted them from physical things.

We could put this doctrine of St. Thomas another way, a way which emphasizes that mathematical quantities are akin to real beings and are not pure entia rationis. All that is positive in the mathematical is found also in its physical foundation.[37] Thus, for example, a mathematical line actually has one dimension, length. Now physical lines also have length plus much more — breadth, width, color, location, motion, sensible matter, etc. As Thomas says, physical things are related to mathematicals by addition for they add sensible matter and motion to mathematical magnitudes.[38] All of these features other than length are separated by abstraction from the mathematical line; yet the length that is retained is itself a real being,[39] though it does not actually exist in reality in the purified condition it has after abstraction. In other words, mathematical quantities as such cannot exist in reality, not because of what they positively have, for that, viz. quantity, does exist in reality, but because of what negatively they do not have — viz. sensible matter, motion, etc.[40] Thus, as many commentators have indicated, mathematical beings for St. Thomas are sort of intermediate, neither purely entia rationis nor real beings, and yet they

share features of both.[41] They are in a way **entia rationis** because *qua* mathematical they exist only in and by the mind's act of abstraction; yet they are in a way real beings since what they are is abstracted real quantity, i.e. real quantity lacking all sensible material features. All of this is present in St. Thomas's statements which we have seen, statements to the effect that the mathematician considers real quantities but not *qua* real; to give but one place where he says explicitly that mathematics (geometry) treats physical quantities:

> Geometry considers the line which has existence in sensible matter, which is the natural line.

And then he adds,

> But it does not consider it insofar as it is in sensible matter, insofar as it is natural, but abstractly, as was said.[42]

Let us note that this passage, and there are many others like it, is in Aquinas's *Commentary on Aristotle's Physics*. This indicates that the position of the two men as to the ontological and mental status of mathematical quantities is fundamentally the same (though St. Thomas does have more detail on the subject).[43] For Aristotle too, as we pointed out in Part I (Chapter II, pp. 63-65), mathematical quantities as such exist only in the mind which has abstracted them; yet these abstracted quantities are actually real quantities whose nonquantitative features have been left aside. (I have referred all along to mathematicals as existing as such in the "mind." Actually this is inaccurate; however, a discussion of the exact "location" of

mathematical quantities will be deferred until Chapter III.)

The Basis of Mathematical Abstraction — the Principle of Dependent Intelligibility

In saying that the mathematician knows real quantities but not as they really exist, it would seem to follow that his knowledge is necessarily erroneous. Now we noted in Part I (Chapter II, pp. 71-72) that Aristotle very clearly states that no error is involved in mathematical abstraction, that it is not erroneous to consider some aspects of a thing apart from its other aspects and from the thing itself. However, we also pointed out that Aristotle never says why this is so, why no error is involved in mathematical abstraction — he merely says that there is none. Aquinas, on the other hand, goes into a great amount of detail on this point. In his Commentary on the Physics,[44] in discussing Aristotle's statement that no error is involved in mathematical abstraction, St. Thomas asserts his agreement with the Stagirite. He then goes on to give two closely related conditions which he says must be fulfilled in order for there to be no falsity in considering as separate an aspect of a physical thing which does not exist separately.

1. The aspect which is considered separate from other aspects must not depend on them for its intelligibility. Obviously, if its intelligibility does depend on them, it can't be understood apart from them — as the nature man cannot be understood apart from animal for it includes animal in its intelligibility. (Here-

after, in this paper, I will refer to this condition as the principle of dependent intelligibility.)

2. The aspect which is considered separate cannot be judged to exist apart from the thing in which it actually does exist.

To relate the first condition to quantity we must recall the doctrine of St. Thomas which we treated in Chapter I of this Part, viz. that the accident of quantity is ontologically prior to all other accidents. In the same Commentary on the Physics, he says,

> [A]mong all the accidents which come to substance, quantity comes first and then the sensible qualities and actions and passions, and the motions consequent upon sensible qualities.[45]

Since this is so, the understanding of quality, action, etc., is not necessary for the understanding of quantity for they are posterior to it.

> [I]t is clear that posterior things are not necessary for understanding prior things, but rather the opposite. Therefore prior things can be understood without the posterior . . . .[46]

And he adds,

> Therefore quantity [which is prior] does not embrace in its intelligibility the sensible qualities or the passions or the motions [which are posterior].[47]

Hence, the first condition is fulfilled: quantity can be considered apart from other accidents even though it does not really exist apart.[48]

In elucidating the second condition, Thomas emphasizes the distinction between the act of judgment and the act of simple apprehension.[49] He asserts that in simple

apprehension, "we can abstract things which are not separated in reality" — so long, of course, as the first condition is fulfilled. However, since

> . . . the second operation has regard to a thing's act of existing (esse), . . . in this second operation [viz. composing and dividing] the intellect cannot truthfully abstract what is united in reality, because the abstraction would signify a separation with regard to the very existence of the thing.

He illustrates using the example of a white man who is musical. One can truly consider the white without the musical, even though they exist together.

> But it would not be a true consideration if one were to assert that the white is not musical.[50]

We see here the notion of the judgment, the second operation, as grasping a thing in terms of its actual existence. Certainly to judge that quantity really exists apart from sensible physical things and all their other accidents would be to say what is not and hence to be in error. But so long as no assertion is made as to its existence separate from physical things and their other accidents, no error is involved in considering it apart from the other accidents since it is ontologically and therefore intelligibly prior to them.

We have pointed out that Aristotle gives no reason why quantity can be considered without error apart from other sensible features. We noted further that he never sets forth any general principle stating what can and what cannot be set aside by abstraction and why. He explicitly mentions neither of the conditions St. Thomas stresses. He

never says that quantity is ontologically prior to other accidents nor that it does not depend on them for its intelligibility. True enough, he does assert that it can be known independently (See Part I, Chapter II, p. 72.) and hence is intelligible apart from them; yet he gives no ontological reason why this is so. This point, then, is a definite contribution by Aquinas to Aristotle's philosophy of mathematics[51] and it is a very important contribution for it plays a very important part in subsequent matters as we shall see. Furthermore, the Stagirite never explicitly applies the distinction between simple apprehension and judgment to the abstraction of the mathematician . He does say that it is false to think that quantity is separate from physical things as the Platonists do. (Part I, Chapter II, pp. 56-57) And it is true that he asserts that truth and falsity are proper only to the mental act of composition and division (judgment) and not to the act of simple apprehension.[52] It would follow that since he says no falsity is involved in the mathematical consideration of quantity and that it is false to say that quantity is really separate from physical things that he must mean no composition or division, i.e. no judgment, is involved in mathematical abstraction. In other words, it is only in an act of simple apprehension that quantity is considered separate. However, none of this is explicit in Aristotle.

Thus, it seems that Thomas Aquinas's elucidation of this point, while certainly not opposed to Aristotle and at least in the case of the second condition actually in ac-

cordance with his stated principles, is definitely an advance and completion of the Stagirite's thought, a very important advance as we shall see.

Use of This Principle in Determining the Constitution of the Mathematicals

We have shown that, according to St. Thomas, the accident of quantity, even though it does not exist apart, can be considered apart from all the other accidents of a physical thing, and hence from what he calls sensible matter. Does this mean that the mathematician considers quantity alone? This seems impossible for as we pointed out in Chapter I of this Part, according to St. Thomas, quantity, insofar as it is by its very nature divisible, demands matter, the principle of potency. In other words, quantity's very intelligibility demands matter. Since this is so, according to Aquinas's principle of dependent intelligibility set forth above, quantity cannot be abstracted from matter — if it does not contain sensible matter then it must possess some other kind. As we might expect, this other kind is intelligible matter. Now we might ask exactly how this matter is present in abstract mathematical objects? Does it play the same role for St. Thomas that it did in the Stagirite's teachings? We will see the answer to this if we ask another (and actually closely related) question — can quantity be considered apart from the material substance in which it actually exists? A moment's reflection will show that this is impossible, again because of the principle of dependent intelligibility. As Aristotle said (Part I, Chapter II, pp.

67-68), and Thomas agrees, substance enters into the very notion and definition of any and all accidents. An accident is that which can neither be nor be understood apart from the substance in which it inheres.[53] This is as true of quantity as of any accident. Thomas explains this in terms of his principle.

> [T]hat form can be abstracted from some matter, the intelligibility of whose essence does not depend on matter of that sort but the intellect cannot abstract form from the sort of matter on which the intelligibility of the essence depends. Consequently, since all the accidents are related to substance as form to matter, and since the nature of every accident is to depend on substance, any accidental form cannot possibly be separated from substance.[54]

We could put this point this way: to know an accident as it truly is, viz. an accident, that whose being is in another, demands knowing it in this other. Thomas says,

> [S]ince the <u>being</u> of the accident depends on the subject, it is necessary that its definition, signifying its being, should contain within it the subject.[55]

(And he goes on to explain that the subject which is contained in the definition of an accident is not there as though it were part of the very essence of the accident, "but as something added to its essence."[56] This point will be of great importance later.)

Recall from Part I that though Aristotle does affirm that accidents must be known in relation to substance, he never applies this principle explicitly to abstracted quantities. Aquinas, on the other hand, clearly makes the application. On the basis of his <u>principle of dependent intelligibility</u>, he states:

> [Q]uantity does not embrace in its intelligibility the sensible qualities or the passions or the motions. Yet it does include substance in its intelligibility.[57]

And in another place he affirms that though the mathematician can abstract quantity from sensible qualities, since it is ontologically prior to them, he cannot abstract quantity from substance.

> Now it is manifest that quantity is in substance before sensible qualities are. Hence quantities, such as number, dimension, and figures, which are the terminations of quantity, can be considered apart from sensible qualities . . . . But they cannot be considered without understanding the substance which is subject to the quantity. . . .[58]

To assert that, according to St. Thomas, substance must be included by addition in the notion of the accident quantity, in other words that quantity in its very intelligibility cannot be abstracted from substance, is not to say that he holds that substance must be included in any very explicit or determinate sense. The intelligibility of quantity does <u>not</u> require that it be understood as the quantity of any particular determinate material substance, for example, that plane surface be understood as plane surface <u>of this desk</u>, or 6 as <u>6 men</u>. Six does not demand for its being or its intelligibility that it be or be known as 6 trees or 6 men. There can be six of any material thing; it is not limited to just one definite kind of substance. It is sufficient, then, that the accidental feature 6 be known as 6 something, that it include this "something" in its intelligibility as its subject but the "something" need be no further specified. Likewise the quantitative feature called

plane surface can be the plane surface of any number of material things, no one of which is therefore necessarily included in its intelligibility.[59] But it must be the plane surface of something and so at least this indeterminate material subject must enter into its intelligible notion. On this point, while affirming that any accident must include in its intelligibility a subject, Thomas states that those accidents

> . . . are said to be simple which have no determinate subject included in their definition, for example, curved and concave and other mathematical entities, and those are said to be compound which have a determinate subject without which they cannot be defined.[60]

He gives snubness as the example of a compound accident whose definition must include a determinate subject, viz. nose (assuming apparently that the only snubbed things are noses). He states, "[N]ose is part of its [snubness's] concept,"[61] just as animal is part of the concept female. He contrasts this with the accident concave which need not include the determinate subject nose in its definition, "since nose is not contained in the concept of concavity."[62] And in the above text, he indicates that he considers accidents of this latter type to be "curved and concave and other mathematical entities." [underlining mine] In other words, mathematical quantities are not accidents which demand a determinate material subject in their definition, though of course like all accidents of physical things, they do demand a material subject.[63]

In connection with this, it should be pointed out

that the mere fact that the mathematician abstracts from all the qualitative and subsequent accidental features does not in my opinion mean that he has thereby left aside the specific formal nature, the particular substantial form, of the quantified substance from which he is abstracting. It is true that by leaving aside all sensible qualities and subsequent accidents he has left out that whereby formal differences <u>are indicated</u> and <u>are grasped</u> by man;[64] nevertheless, the formal difference, the substantial form, is still present, it has not <u>in fact</u> been left out. The analysis in the preceding paragraph of the type of subject which must be included, according to St. Thomas, in the intelligibility of quantity has shown that for him a determinate material subject is not necessary. Hence the particular substantial formal determination of the subject can be left aside in the mathematical abstraction. The basic principle used by St. Thomas in this analysis is the one we set forth earlier and called the <u>principle of dependent intelligibility</u>, viz. one entity can be considered apart from another so long as its intelligibility does not depend on that other. Thus, quantity can be considered apart from any specific material subject since its intelligibility does not depend on any specifically determinate kind of subject. For its being and hence its intelligibility it must be and be known simply as the quantity of some material substance, and this substance need be no further specified. Incidentally, since the specifying substantial form has been left aside by the mathematical abstraction, the determinate quantity that follows upon

a determinate form will also be omitted. For example, an elephant because of the specific kind of thing it is has a maximum and minimum quantity necessarily connected to its nature.[65] Since a mathematical consideration of it solely as a three-dimensional solid would leave aside its nature as an elephant, its determinate maximum and minimum sizes which necessarily follow upon this nature would also be left out. Thus, what would be left as a result of the mathematical abstraction would be a three-dimensional material substance not limited to any determinate sizes and shapes. This point will be important in our later discussion of the freedom of the mathematician in the choice and construction of his quantified objects.

How much of this is present in Aristotle? Of course, as we pointed out in Part I, Chapter II, Aristotle holds that accidents are intelligible only by reference to a subject of their being, and quantity is certainly no exception. Therefore, for him too, quantity can not be abstracted from substance. Yet, as we discussed at length in Part I, Chapter II (section entitled "Partial Solution to Central Problem — the Presence of Substance in Abstracted Quantity), the Stagirite never applies this principle to abstracted mathematical quantities. He never clearly asserts that mathematical quantities must be known in relation to a substance functioning as their substrate. Thomas's explicit statements to this effect are therefore his own contribution. By the same token, Thomas's distinction between accidents which demand a determinate subject in their definition and those which do

not is in his commentary on a passage of Aristotle's Metaphysics where the Stagirite makes the same distinction.[66] Nevertheless, Aquinas's analysis of the material is more detailed and developed than Aristotle's and his explicit relation of this distinction to mathematical quantities is simply not present in the Stagirite. Thus, though Thomas's discussion of the necessary substrate of mathematical quantity rests to a great extent on Aristotelian principles, the explicit development of these principles with particular reference to mathematics is his own achievement. Furthermore, the very important principle which is at the basis of the analysis of Aquinas, what we have called the principle of dependent intelligibility, is, as we said earlier, never made explicit by the Stagirite. Indeed, we have maintained that it is precisely the lack of this principle which is at the root of the Stagirite's other omissions in this area.

Since quantity cannot be abstracted from material substance, according to St. Thomas, and yet this substance need not be determinate, in what way is substance present in the quantitative objects of mathematics? Thomas's position apparently is that it is present in no other way than as the unspecified, nonsensible, almost totally indeterminate material substrate of the accident quantity. And it is at this point that we come upon the matter that must be present in mathematical quantities which by their very nature as divisible demand a material principle. It is here that we find our answer to our earlier question — is St. Thomas's teaching on the intelligible matter which is present in ab-

stract mathematical quantities the same as the Stagirite's?

The Constitution of the Essence of the Objects of Mathematics — Intelligible Matter

It was pointed out in reference to Aristotle's doctrine of intelligible matter that he clearly identifies it with the parts of a mathematical quantity and also clearly says it is the principle of individuation of mathematical quantitative forms and hence their substrate (Part I, Chapter II, pp. 74-75). Though his explanation is not thorough, one meaning of intelligible matter for Aristotle seems to be the substance present as this substrate of the accident quantity. We reasoned that, since the material substance, which must be included in the notion of abstracted quantity, is indeterminate except for its quantitative attribute, he called it matter; since after the abstraction of sensible qualities it cannot be sensed even per accidens but can only be known he called it intelligible. And yet, as we said, the Stagirite himself never explicitly makes the identification of intelligible matter and the substance subject to mathematical quantity. Indeed, he never says in just so many words what intelligible matter is. With St. Thomas Aquinas, on the other hand, there is no question as to what he means by intelligible matter; and there is no question that he identifies it with the substance functioning as substrate of the accident of quantity. In the Summa, for example, he makes an analogy between sensible matter and intelligible matter and states that,

> [S]ensible matter is corporeal matter as subject to sensible qualities, such as being cold or hot, hard or soft, and the like; while intelligible matter is substance as subject to quantity.[67]

(Note, incidentally, that both kinds of matter, sensible and intelligible, have one common element, viz. substance — in one case as subject of quality, in the other only of quantity.)[68] In the De Trinitate he repeats this doctrine.

> [A]ny accidental form cannot possibly be separated from substance. . . [Q]uantity can be considered in substance before the sensible qualities, in virtue of which matter is called sensible, are understood in it. Quantity, then, does not depend on sensible matter with regard to the nature of its substance, but only on intelligible matter. For after accidents have been excluded, substance remains intelligible only to the intellect.[69]

Again in another place in asserting that substance can exist without quantity he explicitly affirms the identity of substance as the substance of quantity with intelligible matter.

> Substance, however, which is the matter of intelligible quantity, can exist without quantity.[70]

And there are other texts which make the same point, substance considered as subject to quantity is intelligible matter.[71]

Incidentally, we should note that it is always substance as subject to quantity that Thomas terms intelligible matter, never substance alone, quantity alone, or substance as subject of quantity and quality.[72] As we have seen, quantity alone is an impossibility, every accident demands a subject, and as we saw in Chapter I of this Part, quantity as divisible demands matter. Substance alone is not present in mathematics but is studied only in metaphysics.[73] Substance as subject to quantity and sensible qualities is what Thomas

calls sensible matter.[74] Hence, substance viewed only as the subject of quantity and not of any other accidents is called intelligible matter by Aquinas.

Of course, as is well known, matter is a term used analogously by St. Thomas as it was by Aristotle; in fact, matter in the primary sense is known only by analogy to secondary matter.[75] In general, any relation of determinable subject to that which determines it can be called a matter–form relation according to Aquinas, and the more determinable an entity is the more material it is.[76] Thus, the elements which make up a brick are as matter to the compound brick itself, the brick is as matter to the wall it makes up, the wall to the entire house; and the advance is from the more material, i.e. less determinate to the less material, i.e. more determinate. The most fundamental kind of physical matter is, of course, prime matter which is completely determinable. The notion of intelligible matter is also used analogously by St. Thomas, as it was by the Stagirite. In addition to substance as subject of quantity, Aquinas refers to the parts of mathematical quantities (the lines of a triangle, the segments of a circle, the units of a number, in general the continuum itself), as matter in relation to the composite whole which they make up.[77] All of these are matter apparently inasmuch as they are that out of which a composite mathematical entity is made, and that into which it can be broken down. They are subjects determinable (potential) by the form (act) of the composite.

Clearly to call all of these matter is to use the term analogously for the units of a number, the lines of a triangle or a circle, and the continuum, are not matter in identically the same sense. For one thing, the discrete units of a number are specifically distinct, according to St. Thomas, from the continuous matter of lines and the continuum.[78] And the continuum itself is more basically matter than lines are for it is more determinable than they and is in fact their matter. Lines, in turn, are matter for figures which, in turn, could be matter for more complex figures. The question arises — what is more fundamentally intelligible matter, substance as subject to quantity or those various elements and parts of mathematical quantities just mentioned? We said that all of these elements are matter apparently inasmuch as they are determinable, are subject to some kind of quantitative determination. And we also said that that which is most determinable is most material. But the most fundamental subject of quantitative determination is, of course, the substance of quantity itself. Lines, the units of number, even the continuum, like all quantities must themselves be present in (and hence determine) a subject. They must be in substance. We have seen repeatedly that this is Aquinas's teaching. Therefore, the subject of these quantities is more basically matter, for it is more determinable, (In fact, it can be determined by any quantity, discrete or continuous.) than they are. It is more worthy of the name intelligible matter. Hence, in the most primary sense, the indeter-

minate substance, the subject of quantity is for St. Thomas the intelligible matter of mathematical quantities. Even the most basic elements in the genus quantity add some determination to substance functioning as their substrate. Thus, a simple quantitative unit, which is not the transcendental unity which adds nothing to being, adds quantitative indivisibility.[79] A point then adds position to this;[80] a line length; a surface width as well as length.[81] None of these elements, therefore, is as material, as indeterminate, as the substance which is the subject they determine. The continuum and the parts of figures and numbers, therefore, are intelligible matter secondarily, for, though they are able to be further determined (and so are material), they already are some quantitative determination of substance.

Inasmuch as we have been speaking of the identity of intelligible matter with the subject of quantity, it would be in order to point out that Thomas Aquinas like Aristotle speaks of this matter as the principle of individuation within mathematical species. As the subject of mathematical quantitative forms, intelligible matter allows specifically the same mathematical form to be in different individual mathematical objects. Thomas states,

> Now that some singulars are considered among the objects of mathematics is clear from the fact that in this order many things of the same species are observed, as many equal lines and many similar figures. . . . Now matter is the principle of individuation not only in singular things but also in the objects of mathematics; for there are two kinds of matter, one sensible and the other intelligible.[82]

Certainly there is no difference between Aquinas and the Stagirite in this regard.

Of course the intelligible matter which is the principle of individuation of mathematical quantities is <u>individual</u> intelligible matter. In Part I, Chapter II (section entitled "Final Solution to Central Problem — the Two Kinds of Intelligible Matter"), we discussed at length that it is only such intelligible matter that Aristotle mentions in connection with mathematics. We pointed out that he seems to limit intelligible matter to individual mathematicals; that he never sets forth a notion of <u>common</u> intelligible matter which would be part of the universal mathematical object and even of the mathematical essence. We should ask, then, in regards to St. Thomas's doctrine — does he have a notion of common intelligible matter and, if so, is it part of the mathematical essence?

In answer to the first question, Aquinas teaches in no uncertain terms that common intelligible matter is present in the universal object of mathematics. To illustrate, the mathematician does not study this singular triangle or that one, rather he is concerned with triangle, the universal. Now to obtain such universal objects, the mathematician abstracts from individual intelligible matter, intelligible matter under the determinate dimensions of this or that individual figure. The type of intelligible matter present in his universal object, therefore, is not individual, but what Thomas calls non-designated, or common, in-

telligible matter. The following rather lengthy passage of Aquinas makes these points:

> There are two kinds of matter from which abstraction is made: intelligible matter and sensible matter . . . . Each, however, can be taken in two ways: as designated and as not designated. I call matter designated if it is considered together with the determination of its dimensions, that is, with these or those dimensions. I call it not designated, however, if it is considered without the determination of its dimensions. In this connection, it must be noted that designated matter is the principle of individuation, from which every intellect abstracts inasmuch as it is said to abstract from the here and now. The intellect of the natural philosopher however, does not abstract from non-designated sensible matter; for it considers man, flesh and bone, in whose definitions non-designated sensible matter is included. The intellect of the mathematician, however, abstracts entirely from sensible matter, though not from nondesignated, intelligible matter.[83]

The mathematician, he points out, abstracts from all sensible matter, both common and individual (designated). He also abstracts from individual intelligible matter, that is, for example, from the particular dimensions of this triangle. Yet he cannot abstract from common intelligible matter. To attempt to abstract mathematical objects from common intelligible matter, St. Thomas says, would be to attempt to consider quantity without considering the substance which must be its subject, and, of course, this is impossible as we have seen for it violates the principle of dependent intelligibility.

> [Q]uantities, such as number, dimension, and figures, which are the termination of quantity, can be considered apart from sensible qualities, and this is to abstract them from sensible matter. But they cannot be considered without understanding the substance which is subject to quantity, for that would be to abstract them from common intelligible matter.[84]

In other words, common intelligible matter must be included in the universal object of mathematics because intelligible matter fundamentally is, as we have said, substance functioning as the necessary subject of the accident of quantity.

But is this common intelligible matter part of the very essence of the objects of mathematics? In order to answer this question we must first clarify what we mean by the object of mathematics. On the very first page of this Part of the dissertation, we noted that St. Thomas holds that the mathematician studies quantitative being, or more precisely being solely in terms of its quantitative attributes. In speaking of the object of mathematics we should distinguish, then, between: (1) the total object which is grasped as the result of mathematical abstraction, viz. quantified substance, and (2) the precise feature of this total object which is studied, viz. its quantity. Both of these must be kept in mind in talking about the essence of the object of mathematics. In asking about this essence, we could refer either to the essence of the total abstracted object or to the essence of only that feature of it which the mathematician studies, viz. the essence of quantity.

As for the essence of quantity, we can definitely say that the common intelligible matter which Thomas identifies with the substance which is the subject of quantity is not an ingredient. This substance, this common intelligible matter, though it is part of the total abstracted object, part even of the essence of that total object, part

also of the definition of quantity, is only added on to the essence of quantity itself. We noted above (pp. 214-215) that Aquinas, like Aristotle before him, spoke of the definitions of accidents as definitions by addition. The subject which must be contained in the definition of an accident is nevertheless not present as though it were part of the essence of the accident, "but as something added to its essence."[85] Thus, for St. Thomas just as for Aristotle common intelligible matter meaning the substance, the subject of quantity, would not be part of the essence of the object of mathematics where by object we mean abstracted quantity. On the other hand, this common intelligible matter would be part of the total abstracted object, quantified substance, and of its essence.

However, intelligible matter in the secondary sense, meaning the parts of a mathematical quantity, would be of the very essence of abstracted quantity. It is the very nature of quantity to have parts distinct from parts; indeed, quantity is described by St. Thomas as consisting "in a certain multiplication of parts."[86] Clearly then, such parts, such intelligible matter, cannot be omitted from its essence. Now this same point was made in reference to Aristotle, namely that intelligible matter in the sense of the parts of mathematical quantities is a necessary ingredient of the essence of such quantities. Yet we also said that the Stagirite himself never draws this conclusion; he never refers to this or to any intelligible matter as part of the essence

of abstracted quantity. In like manner, while Aquinas is very clear that intelligible matter is part of the abstracted mathematical, that common intelligible matter, meaning the substance of quantity, is part of the universal abstracted object, he never says in so many words that common intelligible matter in the other sense, meaning the parts of mathematical quantity, is an ingredient of the essence of such quantities. Nevertheless, since quantity as he describes it is itself no more and no less than a unity of many quantitative parts, such parts, such intelligible matter, must be included in its essence. To take simple examples, a triangle is of its very nature one figure made up of three lines as its parts; likewise a number by its very nature is composed of units. To speak of the essence of a triangle without its three sides or of a number without its units would be absurd.

Conclusion

Thus, we have set forth the precise nature of the object of the science of mathematics according to St. Thomas Aquinas. In briefly summarizing his teachings, I will at the same time indicate his differences with Aristotle. Thomas adopts as his own the Aristotelian doctrine of mathematical abstraction. For both men, the mathematician studies quantities abstracted from quantified physical things. These abstracted quantities which exist as such only in the mind are nevertheless real quantities whose nonquantitative accidents have been left aside. To put it precisely, mathematics studies real quantities but not qua real. There are no basic

differences between the two philosophers on this point though Thomas does have more detail, especially as regards mathematicals as akin to entia rationis having their proximate foundation in the intellect. (For a detailed discussion of this point, see above, pp. 204 ff.)

However, when we come to the ontological justification for mathematical abstraction, i.e. why quantity can be considered separate even though it is not really separate from physical things, Thomas offers a definite contribution to the Stagirite's teaching, and it is an extremely important contribution in determining the exact object of mathematics. The Thomistic doctrine that abstraction of one entity from another is possible even when they are joined in reality so long as the intelligibility of the one abstracted does not depend on the other, is simply not explicitly present in Aristotle, though it is not denied by him. This principle (which I have called the principle of dependent intelligibility), which explains why quantity can be considered apart from all other accidents, also explains why quantity as an accident can be considered apart from a determinate substance but not from indeterminate substance as its subject, in other words, not apart from intelligible matter in the sense of substance as subject to quantity. Because of this principle, we see clearly that the total abstracted object of mathematics for Aquinas is not simply quantity, but quantified intelligible matter. Since this very important principle of dependent intelligibility is

absent from Aristotle's philosophy of mathematics, it is no wonder that his doctrine is at times vague and incomplete concerning the exact nature of mathematical quantities, the nature of intelligible matter, and the nature of mathematical objects as including or excluding such matter.

Of course, as I have said, the entire notion of intelligible matter as an ingredient of the objects of mathematics is much more developed by Aquinas. Not only does he clearly identify it in one sense with substance, the subject of abstracted quantity (something Aristotle never explicitly does), he also just as clearly expounds a notion of common intelligible matter which is part of the universal mathematical object. The Stagirite, on the contrary, seems to have never conceived of a common intelligible matter, again probably because he lacked the principle of dependent intelligibility.

Finally, concerning the question of the essence of the objects of mathematics, we find somewhat more development in Aquinas than in Aristotle. Taking as the "object" of mathematics the total abstracted object, quantified substance, we can say that for St. Thomas common intelligible matter, meaning the substance subject to abstracted quantity, is definitely part of its essence. (As we said Aristotle apparently has no such matter.) On the other hand, taking as the "object" of mathematics the precise aspect under which the mathematician studies the total object he abstracts, viz. as quantified, we can just as definitely say that for St.

Thomas such intelligible matter is not part of its essence. Though it must be included in the definition of quantity, the substance which is its subject is not part of its essence (On this point St. Thomas and Aristotle agree.); hence neither is the common intelligible matter which is identical with it. However, intelligible matter meaning the parts of mathematical quantities must definitely be ingredients of the essences. Unfortunately, this last point, whether or not intelligible matter in its secondary sense is part of the essence of mathematical quantity as such, is no more discussed by Aquinas than it was by the Stagirite. Nevertheless, the common notion of quantity which each man has would seem to demand an affirmative answer.

FOOTNOTES

[1]In XI Metaphysics, L. 3, 2202.

[2]In V Physics, L. 1, 412.

[3]For a more detailed discussion of these characteristics of mathematicals, see Part I on Aristotle, Chapter II, pp. 20 ff. Since Aquinas's doctrine on this point is the same as the Stagirite's, it seems unnecessary to repeat that analysis here.

[4]In VIII Metaphysics, L. 5, 1760.

[5]The following is a list of places where St. Thomas refers to mathematical objects as immobile:

De Trinitate, V, 4, ad 7; VI, 1, 2c.
In I Metaphysics, L. 13, 202; In II Metaphysics, L. 5, 336; In III Metaphysics, L. 4, 375, L. 7, 420, 422; In V Metaphysics, L. 16, 989; In VI Metaphysics, L. 1, 1160-61; In IX Metaphysics, L. 1, 1774; In XI Metaphysics, L. 7, 2260-61.
S. T., I, 5, 3, ad 4.
In III De Caelo, L. 3 (Parma, XIX, pp. 150-51).
In II Physics, L. 3, 160-61, 163.

I know of no place where he calls them unchangeable.

[6]See his Commentary on the Physics, V, L. 2 and 3.

[7]S. T., I, 19, 1 ob. 3.

[8]One or more of these expressions is used in the following places:

In III Physics, L. 12, 392.
In V Metaphysics, L. 14, 974, L. 16, 990-991.
C. G., I, 20, 3.
In V Metaphysics, L. 21, 1093-94.

[9]One or more of these expressions is used in the following places:

C. G., I, 20, 3.
In III Metaphysics, L. 12, 497; In V Metaphysics, L. 14, 974; In IX Metaphysics, L. 1, 1774.
In I Physics, L. 9, 66.
S. T., I, 7, 3 ad 1.

[10]Principles of Nature, #15.

[11]In In V Metaphysics, L. 14, 974 and L. 17, 1024, Thomas speaks of these changes in mathematicals as changes only in an analogous sense. See Maziarz, The Philosophy of Mathematics, p. 207.

[12]In V Metaphysics, L. 16, 992.

[13]Some Thomists hold that mathematicals are completely unchangeable according to St. Thomas. Simon, ". . . Mathematical Abstraction," 122 and 131 held this as does Smith, St. Thomas on the Object of Geometry, p. 14; yet the latter at the same time refers to the making of mathematicals by division and multiplication, p. 42, which seems rather inconsistent. Of course, the point and the unit, since they are completely indivisible, cannot undergo substantial change by being divided — though they do come to be and cease to be. (In III Metaphysics, L. 12, 497, In VIII Metaphysics, L. 4, 1746) Moreover, there are very definite limitations on the way in which mathematicals can come to be from or become other mathematicals. (See Part I, footnote 38 of Chapter II, where this is discussed at length.) (In III Metaphysics, L. 13, 510-11, L. 12, 497) For example, a single mathematical plane cannot be divided in depth to produce two planes, for a mathematical plane has no depth. It can be divided in length and width, however.

[14]S. T., III, 77, 2 ad 4. De Koninck, "Abstraction from Matter . . .," 168-172 and Gabriel, "Matière intelligible . . .," #2, 187-8.

[15]De Trinitate, V, 4c; In II Physics, L. 3, 161; In XI Metaphysics, L. 7, 2260.

[16]One place where Thomas makes this distinction is De Trinitate, V, 1c:

". . . the speculative sciences are distinguished according to their disposition (ordinem) with reference to separation from matter and motion.

"Now there are some objects of speculation which depend on matter with respect to their existence, for they can only exist in matter. And there is a distinction among these. Some depend on matter both with respect to their existence and their concept. This is the case with those whose definition contains sensible matter and which, therefore, cannot be understood without sensible matter; as, for instance, it is necessary to include flesh and bones in the definition of man. Physics or natural science studies things of this sort. There are some objects of speculation, however, which although depending on matter with respect to existence, do not depend on it with respect to their concept, because sensible matter

is not included in their definitions. This is the case with lines and numbers — the sort of things mathematics studies."

Other texts which make this same distinction are: In I Physics, L. 1, 2, 3; In I De Caelo, L. 1, 7 (Parma, XIX, p. 3); In XI Metaphysics, L. 7, 2260. See also De Trinitate, V, 2c. See Maritain's discussion of this, The Degrees of Knowledge, pp. 34 ff.

[17] In VII Metaphysics, L. 9:

"The divisions of a circle [e.g. its segments], however, are not parts of a circle taken formally, but of this part of a circle, or of these circles, as the matter in which the form of a circle is produced." (1474)

"For not all the parts into which a thing is corrupted, when it is dissolved, must be parts of its substance; because even if a line when divided is dissolved into two parts, or a man into bones, sinews, and flesh, it does not therefore follow, if a line is thus composed of halves, or a man of flesh and bones, that these are parts of their substance; but these things are constituted of these parts as their matter." (1476)

See also L. 9, 1480, and L. 10, 1483; In IX Metaphysics, L. 1, 1774; De Trinitate, V, 3, ad 4; and In II Posterior Analytics, L. 9, 5:

"For mathematics abstracts indeed from sensible matter but not from intelligible matter, as is said in Metaphysics VI, which intelligible matter is considered accordingly as something divisible is taken either in numbers or in continua. And therefore, whenever in mathematics something is demonstrated of the whole through the parts, there is seen to be a demonstration through the material cause, for the parts are to the whole according to the reason of matter, as is had in Physics II."

See also Gabriel, "Matiere intelligible . . .," #2, 199 and 208 and John J. Morrison, A Definition of Mathematics, unpublished doctoral dissertation (Notre Dame, 1951), p. 106. Incidentally, the very fact that St. Thomas speaks of the parts of mathematicals as their matter indicates that these parts are that out of which a composite mathematical is formed and into which it can be broken down. In other words, it indicates that mathematicals are generable and corruptible and so can change "substantially." (On this point, see the discussion in Part I, Chapter II, pp. 30-31 and footnotes thereon.)

[18] Quaestiones Disputatae De Potentia, Q. IX, art. VII, c (Parma, VIII, p. 191). The translation is my own.

[19]In XI Metaphysics, L. 7, 2261. See also De Trinitate, V, 1c; In I De Anima, L. 2, 28; In I Physics, L. 1, 2, 3; In II Physics, L. 3, 161.

[20]In I De Anima, L. 2, 28.

[21]In II Physics, L. 3, 160-61.

[22]De Trinitate, V, 3, ad 2. (See also In I De Caelo, L. 1, 7 Parma, XIX, p. 3 and S. T., I, 7, 3c.)

[23]S. T., I, 78, 3, ad 2.

[24]All of the following commentators discuss the difference between abstracted mathematical quantities and common sensibles (real quantities): De Koninck, Preface to Précis de Psychologie Thomiste by Stanislas Cantin, (Quebec, 1948), p. xliv and "Abstraction from Matter, II," 58; Maziarz, The Philosophy of Mathematics, pp. 198-200; Simon, ". . . Mathematical Abstraction," 132; Kane, ". . . distinction of the sciences," 60; Maritain, The Degrees of Knowledge, pp. 53-55, 144.

[25]S. T., I, 78, 3, ad 2; In III De Anima, L. 1, 577-78.

[26]The following commentators all stress this point: Maritain, The Degrees of Knowledge, pp. 53-55 and Philosophy of Nature, pp. 28-30; Mahoney, The Notion of Quantity . . ., p. 16; Leroy, "Le savoir spéculatif," 330-331; Greenwood, Études sur . . ., p. 83; Simon, ". . . Mathematical Abstraction," 135-139.

[27]In I Sententiarum, D. II, Q. I, art. III, c (Parma, VI, p. 23).

[28]Cf. Maurer, "A Neglected Thomistic Text on the Foundation of Mathematics," Medieval Studies, XXI (1959), 187.

[29]In I Sententiarum, D. II, Q. I, art. III, c (Parma, VI, p. 23).

[30]Ibid.

[31]Maurer, "A Neglected Thomistic Text . . .," 190.

[32]In IV Metaphysics, L. 4, 574.

[33]Maurer, "A Neglected Thomistic Text . . .," 190.

[34]Logical intentions which also have their proximate foundation in the intellect do not differ from mathematicals in this respect. However, the remote foundation is different

in each case. In the case of mathematics, the remote foundation of abstracted quantity is the accident of quantity possessed by physical things; in the case of logical intentions it is the unity in reality of diverse rationes. Cf. Maurer, "A Neglected Thomistic Text . . .," 192.

[35]S. T., I, 5, 3, ad 4.

[36]S. T., I, 44, 1, ad 3.

[37]Morrison, A Definition of Mathematics, pp. 97-99, has a very good discussion of this point.

[38]In I De Caelo, L. 3 (Parma, XIX, p. 7):

"[C]orpus naturale addit materiam sensibilem super magnitudinem mathematicam. . . ."

In III De Caelo, L. 3 (Parma, XIX, pp. 150-51):

"[M]athematica dicuntur per abstractionem a naturalibus; naturalia autem se habent per appositionem ad mathematica: superaddunt enim mathematicis naturam sensibilem et motum, a quibus mathematicalia abstrahunt: . . ."

[39]I take it this is what Vincent Smith means when he states that, for St. Thomas, geometry is "not a study of an ideal order but a science of the real world," p. 65 of St. Thomas on the Object of Geometry. Gredt says too that mathematical quantity is real being, cited by Maurer, "A Neglected Thomistic Text . . .," 186. On the other hand, Simon, ". . . Mathematical Abstraction," states categorically, "Mathematics is by no means an ontology of real quantity," 138. Actually, I do not think these positions are necessarily opposed. Simon is stressing that it is not as real that quantities are considered in mathematics. Smith and Gredt are stressing that it is real quantities, however, which are considered. Each is simply stressing a different side to Aquinas's teaching, which is that mathematics does consider real quantities but not qua real.

[40]Morrison, A Definition of Mathematics, pp. 98-99.

[41]Simon, ". . . Mathematical Abstraction," 138-139 and footnote 23, 135-36 which contains John of St. Thomas's position. Vincent Smith, pp. 84-85 of St. Thomas on the Object of Geometry; Morrison, A Definition of Mathematics, pp. 97-99, 103; Mahoney, The Notion of Quantity . . ., p. 16; Maziarz, The Philosophy of Mathematics, pp. 204-207; Maritain, The Degrees of Knowledge, pp. 145-146 and Philosophy of Nature, pp. 26-31.

[42]In II Physics, L. 3, 164.

[43]Greenwood, Études sur . . ., p. 73, points out that Thomas is more explicit than Aristotle on the distinction between mathematical being and real being. This is true, but it indicates no fundamental difference in doctrine in my opinion.

[44]In II Physics, L. 3, 160-61.

[45]In II Physics, L. 3, 160.

[46]In II Physics, L. 3, 161. (Kocourek's translation in An Introduction to the Philosophy of Nature, p. 111)

[47]In II Physics, L. 3, 161.

[48]See also De Trinitate, V, 3c where he makes the same point. For a discussion of this point by modern commentators, cf. Simon, ". . . Mathematical Abstraction," 137-9; De Koninck, Preface to Précis de Psychologie Thomiste, p. xliv; Éleuthère Winance, "Note sur l'abstraction mathématique selon saint Thomas," Revue Philosophique de Louvain, LIII (1955), 485, 495-6; Owens, . . . Christian Metaphysics, pp. 168-170; Maurer, Introduction to Commentary on the De Trinitate . . ., pp. xvi-xx.

[49]All of the following in my text is from De Trinitate, V, 3c. See also In III Metaphysics, L. 7, 422 for a parallel text.

[50]In II Physics, L. 3, 161. See also In III De Anima, L. 7, 781-2.

[51]Geiger, in "Abstraction et separation . . .," stresses at some length that this point is Thomas's contribution, 34-36.

[52]In III De Anima, L. 6; In VI Metaphysics, L. 4.

[53]S. T., I, 85, 1, ad 2. Owens, . . . Christian Metaphysics, p. 170; Geiger, "Abstraction et separation . . .," 33-36; Morrison, A Definition of Mathematics, pp. 25-27; Winance, "Note sur l'abstraction mathématique . . .," 495-6.

[54]De Trinitate, V, 3c.

[55]In I Posterior Analytics, L. 10, 4. Parallel texts are in In VII Metaphysics, L. 4, 1352; Quaestiones Disputatae De Potentia, Q. VIII, art. IV, ad 5 (Parma, VIII, p. 176).

[56]In VII Metaphysics, L. 4, 1352-54.

[57]In II Physics, L. 3, 161. See also S. T., I, 85, 1, ad 2.

[58] S. T., I, 85, 1, ad 2.

[59] In XI Metaphysics, L. 3, 2202 shows that the geometer abstracts from the determinate kind of material thing:

"[T]he science of mathematics carries on its investigations by removing from the scope of its study all sensible qualities, . . . and retains as its object of study only the quantified and the continuous, ... And it is primarily interested in the properties of these inasmuch as they are continuous and not in any other respect; for it does not consider the properties of surface inasmuch as it is the surface of wood or of stone. Similarly it considers the relationships between its objects."

[60] In VII Metaphysics, L. 4, 1343.

[61] In VII Metaphysics, L. 4, 1344.

[62] Ibid.

[63] In In I De Anima, L. 2, 28, he states that "the mathematician abstracts from this or that particular material factor in order to attend exclusively to the purely quantitative." And in In VIII Metaphysics, L. 5, 1760, he says that the mathematician abstracts from "sensible qualities or differences." Both of these texts, along with In XI Metaphysics, L. 3, 2202 (text in footnote 59), seem to indicate that the mathematician abstracts from the specific form of material things.

[64] Owens makes this point in . . . Christian Metaphysics, p. 170.

[65] In I Physics, L. 9, 66; S. T., I, 7, 3c.

[66] Metaphysics, VII, 5.

[67] S. T., I, 85, 1, ad 2.

[68] Gabriel points this out, "Matière intelligible . . .," #2, 188.

[69] De Trinitate, V, 3c.

[70] Ibid.

[71] In II Physics, L. 3, 161; In IV Sententiarum, D. XII, Q. I, art. I, Quaestiuncula III, Solutio III, c (Parma, VII, p. 655). On the identification of intelligible matter with substance as the subject of quantity, see the excellent articles on this topic by Gabriel, "Matière intelligible . . .," #2, 187-199, Geiger, "Abstraction et separation . . .," 36-38, and Simon, ". . . Mathematical

Abstraction," 131-132. The latter, in my opinion, expresses St. Thomas's position perfectly in stating:

"He [St. Thomas] understands intelligible matter to be substance considered precisely and exclusively as that which lies under quantity and without which quantity can neither exist nor be understood."

Other modern Thomists who have seen this identity are Owens, . . . Christian Metaphysics, p. 170; Greenwood, Etudes sur . . ., p. 76; Maziarz, The Philosophy of Mathematics, p. 183, footnote 87; and Edward D. Simmons, The Thomistic Doctrine of Intellectual Abstraction for the Three Levels of Science: Exposition and Defense (Ann Arbor, Mich., 1954), pp. 70 and 149.

[72]Gabriel, "Matière intelligible . . .," #2, 187-8, 193.

[73]De Trinitate, V, 3c.

[74]In VIII Metaphysics, L. 5, 1760 (text on pages 198-199 of this chapter). De Trinitate, V, 3c.

[75]In I Physics, L. 13, 118.

[76]Principles of Nature, #2, #5, #13, #14, #30. See also his lengthy commentary on the principles of nature in In I Physics, L. 12 and 13. See the discussion of Collingwood, Philosophy of Nature, Chapter 6, "Causality of Form and Matter," especially pp. 157 ff. — "Matter and Form Are Correlatives."

[77]In VII Metaphysics, L. 11, 1507; In VIII Metaphysics, L. 5, 1760; In III De Anima, L. 8, 708-714; S. T., I, 85, 1, ad 2; In II Posterior Analytics, L. 9, 5 and texts cited in footnote 17 of this chapter. See also Winance, "Note sur l'abstraction mathématique . . .," 500-504, and Gabriel, "Matière intelligible . . .," #2, 197-198.

[78]In V Metaphysics, L. 15, 978:

"[T]here are two primary kinds [of quantity]: plurality or multitude, and magnitude or measure."

This, incidentally, is why it would be erroneous to simply identify intelligible matter with the continuum. At the most, the continuum is the matter of continuous quantities and it is only as such that St. Thomas refers to it as intelligible matter. Cf. In VII Metaphysics, L. 11, 1507-8; In VIII Metaphysics, L. 5, 1760; In VII Metaphysics, L. 10, 1496; In III De Anima, L. 8, 708, 714.

[79]See footnote 22 of Chapter I of this Part.

[80]In I Posterior Analytics, L. 41, 4:

"[P]oint adds position over and above unity."

[81]In XI Metaphysics, L. 3, 2202; In V Metaphysics, L. 13, 949.

[82]In VII Metaphysics, L. 10, 1494, 1496. A parallel text is In III De Anima, L. 8, 708. See also Owens, . . . Christian Metaphysics, p. 170; Geiger, "Abstraction et separation . . .," 37-38; Maziarz, The Philosophy of Mathematics, p. 207 and p. 216, footnote 42; Morrison, A Definition of Mathematics, pp. 43, 53, 259.

[83]De Veritate, 2, 6, ad 1. Cf. also S. T., I, 85, 1, ad 2.

[84]S. T., I, 85, 1, ad 2.

[85]In VII Metaphysics, L. 4, 1354. See the whole of his discussion of the definition of accidents 1352-55.

[86]C. G., I, 69.

CHAPTER III

THE ROLE OF IMAGINATION IN MATHEMATICS

Whereas Aristotle never explicitly referred to the imagination in connection with mathematics or its object, St. Thomas's teaching is very explicit — the imagination has an extremely important role to play in this area. Indeed, as we shall see, the fact that the objects of mathematics are imaginable is used by Aquinas as one way of distinguishing this science from metaphysics and physics.

Now, as is well known, Thomas follows Aristotle in asserting the absolutely essential role of the imagination in all of man's knowledge. We noted in Part I (Chapter III) the essential role of "memory" set forth by the Stagirite in his famous text on abstracting the universal in the Posterior Analytics. St. Thomas in his commentary on this passage[1] repeats with approval Aristotle's teaching on the function cf "memory," viz. to provide a fundamental stability in the mutable objects grasped in direct sensation, from which stable objects experience, and thence knowledge of the universal, can arise. In his commentary on this passage, Thomas follows its author in using, not the term imagination, but the term memory. However, it is very clear in other places that for Aquinas it is actually imagination that

plays this very necessary role of providing the stability in sense knowledge, in other words of providing the proper sense image from which the intellect may abstract the universal.[2] The following text shows clearly his teaching on these points.

> The image [which is the product of the imagination] is the principle of our knowledge, as that from which the operation of the intellect begins, not as something fleeting, but enduring as a sort of foundation of intellectual activity. . . .[3]

Since there can be no intellectual knowledge at all without the phantasms provided by the imagination,[4] we see clearly the indispensable role this faculty plays for all knowledge.

Though it is absolutely necessary for <u>all</u> knowledge, imagination has a special function in the science of mathematics for St. Thomas. He, unlike Aristotle, leaves no question that mathematicals, some at least, are imaginable. His texts which assert this are numerous; I will cite only one:

> When sensible characteristics are removed there remains something which is apprehended by the imagination. . . . Now mathematicals are of this sort.[5]

Of course, in speaking of mathematicals being grasped by the imagination, Thomas is referring not to mathematical essences — these are grasped only by the intellect — but to <u>individual</u> mathematicals. We noted in Chapter III of Part I (page 108) that Aristotle speaks of individual mathematicals as grasped by "intuition" while mathematical essences, he says, are grasped by the mind; and he defines this intuition no further. St. Thomas interprets it as

meaning imagination,[6] and, of course, it is very logical that he would. Individual mathematicals as such are not attained by external senses, nor are they as individual present in the intellect which is directly of the universal.[7] Yet as individual they must be grasped by a sense power. The internal sense of imagination would, therefore, be their logical "location." Aquinas very clearly says that it is the individual mathematicals which are grasped by imagination.

> [S]ome singulars are considered among the objects of mathematics. . . . And such singulars are said to be intelligible insofar as they are grasped without the senses by means of imagination alone, which is sometimes referred to as an intellect.[8] [underlining mine]

and he clearly distinguishes that which the imagination grasps from the essence of mathematicals which is grasped by the intellect.

> In the case of mathematics, it can be shown that that which knows the essence, i.e. the intellect, is distinct from what apprehends mathematical objects themselves, i.e. the imagination.[9]

Now since individual mathematicals are imaginable and yet are not directly sensible, for they lack sensible qualities, the imagination and only the imagination is able to provide the proper object from which the intellect can abstract the universal mathematical essence. This point deserves elaboration. Mathematical quantities, because they are abstracted, lack all proper sensible qualities — hence they cannot be grasped by the exterior senses; yet they are not purely intelligible like the objects of meta-

physics, they are imaginable, says St. Thomas. Now the direct senses are able to supply an appropriate object for the abstraction of physical essences; the imagination simply provides a stability in the changing objects grasped by sense. But the direct senses themselves cannot provide an appropriate object for abstraction of mathematical essences, for mathematical objects as such are not grasped by the exterior senses as we have said; therefore, the imagination can not just provide stability in objects grasped by them if it is to supply appropriate mathematical phantasms to the intellect. Rather the imagination itself, and it alone, must in some way provide the proper object, the abstracted individual mathematical quantity, from which the mathematical essence can in turn be abstracted. Thus, we see the extremely crucial role imagination would have in mathematical abstraction for St. Thomas; only it can provide a suitable phantasm as the basis for the abstraction of the mathematical universal.

Of course, to say that individual mathematicals are imaginable seems to present as many problems as it solves. We discussed these problems in Part I (Chapter III, p. 107) when we attempted to give reasons why Aristotle <u>never</u> says that mathematicals are imaginable. It might be beneficial to repeat these reasons here.

The imagination is a sense power which has for its object the sensible as absent inasmuch as it retains and can recall a sensible object even when it is no longer pres-

ent.[10] Thomas agrees with Aristotle that imagination has for its object what is sensible, that its images differ from sensible objects only insofar as they are not material.[11] But mathematicals are not sensible objects; in his abstraction, the mathematician leaves aside sensible qualities. How, then, can nonsensible mathematicals be grasped by a sense power, the imagination?

Because of this difficulty, one might be tempted to say that the mathematicals which St. Thomas designates as imaginable are not really the individual qualityless, non-three-dimensional, objects of mathematics but only individual objects which come very close to being like them, e.g. a colored line made up of very, very small dimensions, but not actually colorless or unidimensional. According to this view, the individual qualityless unidimensional lines studied in geometry would not really exist at all. Since they do not actually exist as mathematical in physical things, if they are not in imagination they simply would not be at all. Only their abstract nature in the intellect would actually be.[12] However, this interpretation is contrary to many explicit statements by Aquinas. He states in no uncertain terms that mathematical objects (in other words, the qualityless, uni and bidimensional entities) are in the imagination. For example, in the De Trinitate he asserts:

> [M]athematicals themselves come under the senses and are objects of our imagination, such as figures, lines, numbers and the like.[13]

Another equally explicit text is from the De Veritate:

> There is imagination for all bodies insofar as they share in the character of quantity, figure, and the things which follow on this. Hence, it [imagination] extends not only to physical objects, but also to mathematical objects.[14]

And there are many more places where Aquinas makes the same assertion.[15] Furthermore, he states unequivocally that both the <u>particular</u> as well as the universal mathematical notions abstract from sensible matter, "matter as subject to sensible qualities."[16]

> [M]athematical entities are altogether separated from sensible matter in the understanding, because sensible matter is not included in the understanding of the mathematicals, neither in the universal nor in the particular.[17]

And he asserts in reference to abstracted particular mathematicals that "when sensible characteristics are removed there remains something which is apprehensible by the imagination."[18] In fact, mathematics is the most certain science, according to St. Thomas, precisely because its objects are free from sensible matter and yet imaginable.[19] We are still, therefore, faced with the problem, how can objects lacking sensible qualities be apprehended by a sense power?

It could be suggested in answer that there are other sense powers in St. Thomas's teaching, other internal senses, which are somehow able to attain nonsensible aspects of things. The estimative power which "apprehends intentions not received through the sense" immediately comes to mind, not to mention the memorative power, which preserves the intentions apprehended through the estimative sense.[20] However, in the light of Aquinas's assertions that the sensible is the object of

the imagination, we can hardly consider it a sense power which somehow grasps the nonsensible. Rather, the solution may lie in showing that for St. Thomas mathematicals are in a way sensible and therefore imaginable; in other words, in showing that mathematical abstraction does not leave aside all the sensible attributes of quantified physical things.

To show this, we must return to the part played by the imagination in the process of abstraction. Though the imagination is fundamentally a retainer of sensible objects, it is not limited to simply reproducing these objects as they were originally sensed. St. Thomas states that man through this power is able to combine and divide imaginary forms (or in another place he speaks of it synthesizing and analyzing) and thus end up with images of various things, "even of things not perceived by the senses," e.g. a golden mountain which he says is an obvious example of combination on the part of the imagination.[21] Actually the other operation, that of division, is more pertinent to our problem. By this operation, Thomas apparently refers to our ability to separate certain aspects of a given image from other aspects, including the ability to recall not only an entire sensed object but also some part of it divided off from the whole, e.g. to recall only the color of an apple without at the same time its taste. Now bearing in mind that the common sensibles like the proper sensibles are directly, not incidentally sensed,[22] would it not be possible by the operation of division for the imagination (under the guidance of

the intellect, of course — since animals do not have this power, St. Thomas says)[23] to present an image of an originally apprehended physical thing, an image which would be of only part of the thing, viz. of some or all of its dimensions, minus all of the proper sensible qualities of the thing? This image could even have these dimensions reduced to such an extent that in some cases it would be only unidimensional (an image of lines) or bidimensional (an image of planes). (We will speak of nondimensional objects later.) Thus, it seems that the operation of division which St. Thomas attributes to the human imagination allows us to at least give a <u>plausible</u> explanation (and I say this because Aquinas does not himself explain how mathematicals come to be imaginable) of how mathematicals can be sensible and hence imaginable. This operation of the imagination explains how images which are true mathematicals (lines, figures, etc.) can come to be present in this interior sense power. It explains how "even when sensible characteristics are removed there remains something which is apprehensible by the imagination."[24] At least some mathematicals are sensible, and hence imaginable, because they are the abstracted dimensional quantitative features of physical things. However, these imagined dimensions are mathematical and not physical because by this power they have been separated from the other sensible characteristics of physical things and may have even been reduced in dimension from the given physical three dimensions. Also, of course, the imagination could make an

abstracted length perfectly straight even though its physical counterpart may have been curved.

Of course, in order to say that at least some mathematicals are sensible, we must in some way reinterpret those places where Aquinas states categorically that mathematicals are not sensible. In my opinion, in such places he is referring to only the proper sensibles, for though he states time and again that the mathematician abstracts from all sensible qualities, he continually makes it clear that these qualities from which he abstracts are accidents which follow after the accident of quantity. We have seen this point before, to cite just one place:

> Accidents, . . . befall substance in a definite order. Quantity comes first, then quality, then passions and motion. So quantity can be considered in substance before the sensible qualities, in virtue of which matter is called sensible, are understood in it.[25]

Clearly the sense qualities he is talking about, those which follow quantity, are only the proper sensibles. Quantity, however, is a common sensible.[26] Since the mathematician does not abstract from the accidents of quantity according to Thomas neither does he abstract from all sensible features. The dimensional figures studied by mathematicians are not sensible inasmuch as they lack all proper sensible qualities. Since it is "the sensible qualities [which follow after quantity] in virtue of which matter is called sensible,"[27] the mathematicals can be called nonsensible. They are sensible, and hence imaginable, however, inasmuch as they are abstracted dimensions, for dimensions are sensible.[28]

Does this explanation turn the mathematicals into nothing more than imagined common sensibles, i.e. imagined real quantities? Definitely not. For the common sensibles, the quantitative attributes of things, are, according to Aquinas, as we have stressed earlier, mobile, spatial, temporal, colored, tangible, etc. On the other hand, imagined quantity, precisely because it is in the imagination, is freed by the activity of this power from the physical conditions of motion, space, and time and from proper sensible qualities. In other words, the imagination performs an abstraction on the common sensibles; after all, it is not only the intellect which abstracts.[29] Mathematicals are sensible and hence imaginable but they are not to be simply identified with the common sensible attributes of quantity present in natural things.[30]

Incidentally, the fact that he holds that some individual mathematicals are imaginable does not require St. Thomas to modify his basic notion of <u>intelligible</u> matter, i.e. he need not call it instead <u>imaginable</u> matter.[31] Viewing it in its most basic sense as substance subject only to quantity it is still, even as the subject of individual imaginable mathematicals, only able to be grasped by the intellect. "[T]he sense powers," he says, "do not reach a comprehension of substance." Therefore it is still properly called "intelligible."[32] It is true that some of the individual quantitative determinations of substance, which determinations Thomas calls intelligible matter in a secondary sense, are

actually imaginable: e.g. these particular lines, this surface, this continuum, etc. Such features could perhaps more properly be called imaginable rather than intelligible, for as individual they are immediately present in the imagination — not the intellect. However, this is not true of that which is their subject. It is only intelligible.

In saying that mathematicals are sensible and hence imaginable, we should not be understood to mean that every single mathematical is such according to Aquinas. In his Commentary on the De Anima,[33] St. Thomas following Aristotle seems to say that points, which are dimensionless units having position, and units, which are both dimensionless and positionless, precisely because they lack all dimension cannot be grasped by any sense power but are only known mentally by negation.

> Sensibles therefore are the first things intellectually apprehended; and these all have some magnitude; hence the point and the unit can be only negatively defined.[34]

It would follow from this that that which is imaginable, since imagination is a sense power, must have some dimension, though not necessarily all three. Hence, a number, which is a plurality of units, would not be imaginable, though some symbol representing it could be. However, this conclusion contradicts a statement by Aquinas in the De Trinitate in which he includes numbers in his list of imaginable mathematical entities.

> [M]athematicals themselves come under the senses and are objects of our imagination, such as figures, lines, numbers and the like.[35]

Are numbers imaginable or not for St. Thomas? In almost all of the places where he refers to particular imaginable mathematical objects, he lists only entities having dimension, geometrical figures, lines, etc. The text of the De Anima (plus the fact that it seems impossible that any totally dimensionless entity could be sensible) leads me to believe that he was not speaking precisely in the De Trinitate. (Indeed, in this text he also asserts that mathematicals "come under the senses," a statement which would certainly have to be loosely interpreted since they obviously cannot be grasped by external senses.) Thus, it seems most likely that only dimensional mathematicals are imaginable according to Aquinas.[36]

Let us now sum up this point. St. Thomas can say that mathematicals (at least those having dimension) are imaginable because in a way they are sensible. That "way" is this. According to Aquinas, the size, shape and extension, the quantitative features of a natural thing, are, like the proper sensibles, sensible not incidentally but directly. Man by his imagination can abstract, i.e. can represent only certain aspects of these things, viz. the quantitative. By his imagination, he can even abstract to such a degree that the images are of objects consisting of as few as one dimension. This explanation is a way (St. Thomas does not explicitly give his way.) of accounting for the imaginable character of the mathematical quantities which Aquinas affirms. The solution is based on his thought regarding

the nature of mathematicals, of the imagination, and of mathematical abstraction.

All of this can now be put into the more general context of mathematical abstraction. The particular objects from which the abstract mathematical essences ultimately arise are physical things, or more precisely, physical things *qua* quantified. Yet the quantities found in particular natural things have radically different characteristics from the quantities studied in mathematics as we have seen. On the other hand, the particular *imagined* quantities, quantities abstracted by the imagination from physical qualities, motion, etc., are exactly the concrete embodiment of true mathematical natures. Hence, they can aptly serve as the most proximate basis for the abstraction of the mathematical universals. In his *Summa*, Thomas mentions the steps in the abstraction of mathematical natures.[37] Mathematicals, he says, are first abstracted from both individual and common sensible matter (e.g. from the triangular figure existing in nature bound up with all the characteristics of physical things). The mathematical essences (e.g. the nature triangle) are further abstracted from individual intelligible matter (e.g. from individual mathematical triangles present in the imagination).

Thus, as we said earlier, though imagination plays an essential role in all abstraction, it has a special function in mathematical abstraction for Thomas Aquinas — a function never adverted to by Aristotle. For, in addition

to providing a stable image from which the universal can be abstracted (This it does in all abstraction.), in mathematics it furnishes to the intellect perfectly appropriate individual mathematicals, which simply cannot be found in nature, from which the mathematical essence can then be abstracted.

We spoke in the previous chapter about the generation and corruption of mathematicals. Since it is only the individual, not the essence, which is generated or destroyed,[38] and since individual mathematicals exist in the imagination for St. Thomas, this must be where their combining and dividing takes place. In other words, it would be in the imagination that the intelligible matter is formed and reformed into different individual mathematicals.

Speaking of forming and reforming different mathematicals, it is interesting to note that St. Thomas states very clearly that it is by construction that mathematicals come to exist.[39] Since construction must be of the individual, it must take place in the imagination; it would be that whereby individual mathematicals are generated and others destroyed. The fact that mathematicals are generated and destroyed in the imagination naturally raises the question as to the freedom the mathematician has in the constructions of his object. Is he free, according to Thomas, to imaginatively construct any object he pleases, or must they be objects which correspond to quantified physical things? In other words, is he free to form intelligible matter in any

way he pleases? This question of freedom in mathematics according to Aquinas will be discussed at length in the next chapter.

Since individual mathematicals are caused to exist by and in the imagination — both by its act of abstraction and by construction, it is only logical for St. Thomas to say that the imagination is the "location" of the termination of the judgments of mathematics. In the De Trinitate, Aquinas distinguishes between the origin and the termination of man's knowledge.[40] "Now the beginning of all our knowledge," he says, "is in the senses;" however, the termination of knowledge is different in each of the three kinds of science. "[J]udgment in mathematics," he says, "must terminate in the imagination." I take this to mean that these judgments are true of, refer to, imaginable realities. Thomas says, if a judgment is true of realities which are only intelligible, it must stop, terminate, in the intellect as do metaphysical judgments; it could not refer to imaginable or sensible realities and still be true of purely intelligible entities merely qua intelligible. Also he says, if a judgment is true of imaginable realities which are not sensible, it must stop, terminate, in the imagination;

> . . . because, when sensible characteristics are removed there remains something which is apprehensible by the imagination, we must judge about such things according to what the imagination reveals.[41]

Finally, a judgment true of sensible realities must stop in the senses. Thus, since mathematicals according to St. Thomas are neither sensible things of nature, nor purely in-

telligible entities, but some at least are imaginable, a judgment about these objects cannot terminate, "stop," in the senses, nor simply in the intellect, but rather must terminate in the imagination. In order to be true, judgments dealing with imaginable objects must refer to what the imagination presents.[42]

There is another reason why judgments about mathematicals are said to terminate in the imagination. As we have seen, according to Aquinas, in the mental act of judging we grasp the existential act of an object; we grasp an entity as it exists. This is in distinction to the act of apprehension which only grasps the nature of a thing and not its act of existence.[43] Now since some individual mathematicals exist as such by and in the imagination, because of its abstraction and construction, the act of judgment must refer to, terminate in, that which the imagination presents. Of course, if the mathematician's judgments need only refer to imagined entities according to St. Thomas this would seem to make him (the mathematician) very free in his choice of object and the operations he performs on it.[44]

Is there anything in Aristotle similar to Thomas's notion of the termination of a judgment? I can find no explicit mention of such a doctrine; however, as we have seen, the Stagirite does assert that truth and falsity pertain only to the second acts of the mind, the acts of composition and division.[45] Since he says that a proposition (the expression of an act of composition and division) is true because it accurately states what is,[46] it follows obviously

that mathematical propositions will be true if they accurately state the way mathematicals are. But how and where are individual mathematicals for Aristotle? Certainly, if they are in the imagination it would follow that true mathematical propositions must refer to these imaginable objects; or to say the same thing, the mathematical judgment must terminate in the imagination. However, as we have pointed out again and again, the Stagirite never explicitly says that individual mathematicals are in the imagination — though from considering other points in his doctrine, it seems reasonable that he would place them there. (See the discussion of this in Part I, Chapter III, p. 108 ff.)

Thus, with his very explicit teaching on the imaginable character of individual mathematicals, St. Thomas has supplied what seemed lacking in this area in Aristotle's philosophy of mathematics. He has located these individual objects in a sense power and yet, remaining true to the doctrine of abstraction, not in the exterior senses; he has thereby acknowledged the special role the imagination has in mathematics in furnishing the only suitable object from which the mathematical essence may be abstracted by the intellect and has also provided a suitable "place," outside of the sensible world, for the generation and destruction of mathematicals, i.e. for their construction.

Furthermore, in addition to his affirmation that mathematicals are imaginable, Thomas's assertion that mathematical judgments do not terminate in the senses but in the

imagination, seems to clearly liberate the mathematician from the restriction of considering only those mathematical quantities which resemble physical quantities. The conclusions of the science of abstract quantity need not, in fact cannot, "square" with sensible physical quantities but only with the quantified entities the mathematician himself constructs in his imagination. Moreover, since Thomas also holds that imagination under the guidance of the intellect can combine and separate images at will, it would seem to follow that the mathematician would be extremely free in the choice of the nature of his quantitative object. Recall in this connection that Thomas holds that the intellect, not the physical thing, is the <u>proximate</u> foundation of mathematicals, i.e. only through it guiding the imagination does mathematical quantity come to be either by abstraction or construction. It would seem, then, that the mathematician can form the intelligible matter as he so desires, causing this or that definite quantitative object to come to be or cease to be so long, of course, as he does not attempt to make a self-contradictory object like a square circle. It is true that this freedom is never, to my knowledge, explicitly affirmed by St. Thomas; nevertheless, it seems to follow very clearly from the aforementioned principles which he does plainly assert. Aristotle, on the other hand, not only says nothing explicitly about freedom in mathematics, he also never enunciates a relation between mathematicals and imagination which would (and does in St. Thomas) definitely

imply such freedom. Thus, in this area too, Aquinas has significantly contributed to his predecessor's doctrine. More will be said in the following chapter on this question of freedom in mathematics.

FOOTNOTES

[1]*In II Posterior Analytics*, L. 20.

[2]De Koninck, "Abstraction from Matter. . .," 140-1; W. Gerhard, "Natural Science and the Imagination," *Thomist*, XVI (1953), 190-216.

[3]*De Trinitate*, VI, 2.

[4]*S. T.*, I, 85, 1c; *In III De Anima*, L. 12, 781; *De Trinitate*, VI, 2, ad 2. For a good discussion of Thomas's notion of abstraction and the role of sense images, cf. Collingwood, *Man's Physical and Spiritual Nature* (New York, 1963), Chapter 5.

[5]*De Trinitate*, VI, 2c. Other texts which affirm that mathematicals are imaginable are: *De Trinitate*, VI, 1c and 2, ad 2; *De Veritate*, 15, 2c; *In VII Metaphysics*, L. 10, 1495; *In III De Anima*, L. 8, 715-6; *S. T.*, I, 7, 3c; *In III Physics*, L. 7, 341; and *Commentary on the Nichomachean Ethics*, translated by C. I. Litzinger, O.P. (2 vols.; Chicago, 1961), VI, L. 7, 1210, 1214. All further references to the *Commentary on the Nichomachean Ethics* will be to this edition unless otherwise noted.

[6]*In VII Metaphysics*, L. 10, 1494-5:

"[T]hat some singulars are considered among the objects of mathematics is clear from the fact that in this order many things of the same species are observed, as many equal lines and many similar figures. And such singulars are said [by Aristotle] to be intelligible insofar as they are grasped without the senses by means of imagination alone, which is sometimes referred to as an intellect, according to the statement in Book III of *The Soul*: 'The passive intellect is corruptible.'

"Therefore there is no definition of singular circles, because those things of which there is definition are known by their own definition. But singulars are known only as long as they come under the senses or imagination, which is called [by Aristotle] an intellect here because it considers things without the senses just as the intellect does."

[7]*In III De Anima*, L. 8, 711-13.

[8]In VII Metaphysics, L. 10, 1494.

[9]In III De Anima, L. 8, 715. On this point, see Maziarz, The Philosophy of Mathematics, p. 207; Gabriel, "Matière intelligible. . .," #2, 181-2; Smith, St. Thomas on the Object of Geometry, pp. 41-43; De Koninck, "Abstraction from Matter, II," 65-67; Cantin, "L'intelligence selon Aristote," 252-288.

[10]S. T., I, 78, 4c.

[11]S. T., I, 84, 6, ad 2; In III De Anima, L. 6, 657-8; In III De Anima, L. 13, 792.

[12]Some contemporary Thomists do hold this view, viz. that real individual mathematical objects can not actually be present in imagination. They agree that imagination possesses only sensible objects which come close to being like mathematicals. See, for example, Bernard Lonergan, "Note on geometrical possibility," Modern Schoolman, XXVII (1950), 127; Winance, "Note sur l'abstraction mathématique . . .," 509; Collingwood, "Intelligible Matter in Contemporary Science," Proceedings of the American Catholic Philosophical Association, XXXVIII (1964), 110 and Man's Physical and Spiritual Nature, pp. 161-2. Others hold that mathematicals, at least those of Euclidean geometry, are imaginable. Cf. Maurer, Introduction to Commentary on the De Trinitate . . ., p. xxxiv; Maziarz, The Philosophy of Mathematics, pp. 165, 203, footnote 104, p. 223; Maritain, The Degrees of Knowledge, p. 40, footnote 2, p. 131, footnote 2. Yet in another place, the latter author seems to say just the opposite, p. 144.

[13]De Trinitate, VI, 1, 2c.

[14]De Veritate, 15, 2c.

[15]See the texts in footnote 5 above.

[16]S. T., I, 85, 1, ad 2.

[17]In II Physics, L. 3, 162.

[18]De Trinitate, VI, 2c.

[19]De Trinitate, VI, 1, 2c:

"For since mathematics is intermediate between natural and divine science, it is more certain than either. It is more certain than natural science because its investigation is not bound up with motion and matter, while the investigation of natural science centers upon matter and motion. . . .

"[M]athematics is also more certain than the method of divine science, because the objects considered by divine science are further removed from sensible things, from which our knowledge takes its origin. . . . But mathematicals themselves come under the senses and are objects of our imagination, ... So the human intellect, which receives its knowledge from images, knows these things with greater ease and certitude than it does a separate intelligence or even the nature of substance, or act and potency and the like."

De Trinitate, VI, 1, 2, ad 2; In VI Ethics, L. 7, 1210.

[20]S. T., I, 78, 4c.

[21]S. T., I, 78, 4c, I, 84, 6, ad 2; C. G., I, 65.

[22]S. T., I, 78, 3, ad 2:

"Therefore the common sensibles do not move the senses first and of their own nature, but by reason of sensible quality; as the surface by reason of color. Yet they are not accidental sensibles, for they produce a certain diversity in the immutation of the senses. For sense is immuted differently by a large and by a small surface. . . ."

[23]S. T., I, 78, 4c, I, 84, 6, ad 2; C. G., I, 65. For places where he talks about the intellect controlling the imagination, see C. G., II, 73; S. T., I-II, 17, 7c and ad 3; I, 57, 4, ad 3.

[24]De Trinitate, VI, 2c.

[25]De Trinitate, V, 3c. See also S. T., I, 85, 1, ad 2.

[26]S. T., I, 78, 3, ad 2:

"As to size and number, it is clear that they are species of quantity. . . . Now quantity is the proximate subject of the qualities that cause alteration, as surface is of color."

Note in this text that the common sensibles of size and number are identified with accidents which are more basic than the proper sensible qualities. In XI Metaphysics, L. 3, 2202, makes it very clear that the mathematician only "strips away" the proper sensibles by his abstraction, not quantity which is a common sensible. Incidentally, I am not implying that all common sensibles are quantities. In the Summa article cited above, Thomas says merely they are "reducible to quantity." My point here is rather the opposite; quantity in the case of both magnitude and number is a common sensible. But since it is sensible, it is imaginable.

[27]De Trinitate, V, 3c.

[28]Cf. Maritain, The Degrees of Knowledge, p. 144.

[29]To abstract, St. Thomas says, is to consider one entity without another when they are actually together in reality. (De Trinitate, V, 3c) Since each sense power considers only what is proper to it and omits all other features of the material thing, it can truly be said to abstract. Cf. S. T., I, 85, 3, ad 2. See Collingwood's excellent discussion of the role of imagination in mathematical abstraction, especially its action of "negating" — Man's Physical and Spiritual Nature, pp. 160-163.

One should not identify this abstraction of the imagination with the second degree of abstraction else he will end up with the difficulty Winance has, "Note sur l'abstraction mathématique. . .," 507 ff. He clearly sees that merely eliminating sensible qualities by the imagination does not result in an object of a different degree of intelligibility, or indeed in any intelligibility at all. Therefore, because he has identified this abstraction of the imagination with the second degree of abstraction, he denies it any validity as a means of distinguishing the intelligible objects of the sciences, 510. The degrees of abstraction for St. Thomas refer to abstraction by the intellect from matter and motion, De Trinitate, V, 1c.

[30]We have discussed this at length in Chapter II of this Part, pp. 203-4.

[31]Winance, "Note sur l'abstraction mathématique. . .," 508-510, seems to have a difficulty on this point. He points out correctly that simply the abstraction of sensible qualities does not result in an intelligible object. However, he seems to feel that Thomas's use of intelligible here (rather than imaginable) gives the opposite impression. His difficulty is solved, I think, by noting that Thomas's use of the term intelligible matter is analogous (as we have pointed out in the previous Chapter, pp. 222 ff.) and that it refers primarily to that which is only intelligible, namely substance.

[32]In the text in which he clearly identifies intelligible matter with substance as subject to quantity, Thomas just as clearly states why this matter is called intelligible, De Trinitate, V, 3c:

"Quantity, then, does not depend on sensible matter with regard to the nature of its substance, but only on intelligible matter. For, after accidents have been excluded, substance remains intelligible only to the intellect, because the sense powers do not reach a comprehension of substance."

[33]In III De Anima, L. 7, 758.

[34]Ibid.

[35]De Trinitate, VI, 1, 2c.

[36]Both Maziarz, The Philosophy of Mathematics, p. 233, and Gabriel, "Matière intelligible. . .," #2, 209, speak of arithmetical entities as able to be imagined only symbolically.

[37]S. T., I, 85, 1, ad 2:

"Mathematical species, however, can be abstracted by the intellect not only from individual sensible matter, but also from common sensible matter. But they cannot be abstracted from common intelligible matter, but only from individual intelligible matter. . . . [T]hey cannot be considered without understanding the substance which is subject to the quantity, for that would be to abstract them from common intelligible matter. Yet they can be considered apart from this or that substance, and this is to abstract them from individual intelligible matter."

[38]De Trinitate, V, 2c.

[39]In I Posterior Analytics, L. 2, 5 (text given in Chapter IV of this Part, p.275). See also In II Posterior Analytics, L. 6, 4. Many contemporary Thomists speak of the constructural character of mathematics, e.g. Maritain, The Degrees of Knowledge, pp. 54-55, 144-145; De Koninck, "Abstraction from Matter. . .," 135-138; Robert Smith, "The Liberal Arts," 81-82; Maziarz, The Philosophy of Mathematics, p. 140.

[40]De Trinitate, VI, 2c. In I Posterior Analytics, L. 30, 5 also speaks of the origin of all knowledge including mathematical in the senses.

[41]De Trinitate, VI, 2c.

[42]Maritain, The Degrees of Knowledge, pp. 131-32; Maziarz, The Philosophy of Mathematics, pp. 202, 205; Gabriel, "Matière intelligible. . .," #2, 184; Morrison, A Definition of Mathematics, p. 70; Simmons, The Thomistic Doctrine. . ., pp. 148-49, footnote 32.

[43]De Trinitate, V, 3c. On the judgment in St. Thomas, see Owens, . . . Christian Metaphysics, pp. 52-56.

[44]In physics the intellect in its judgments must conform to sensible being as it reveals itself to be. Man does not make it. In metaphysics the intellect must conform it-

self to intelligible being as it is — not as man might like to make it be. In mathematics, however, the intellect need not conform to being as sensible or as intelligible but only to mathematical being, being which exists only in the mathematician's imagination.

[45] In III De Anima, L. 6; In VI Metaphysics, L. 4.

[46] Metaphysics, IX, 10, 1051b 7-9.

CHAPTER IV

MATHEMATICS AS A SCIENCE

Mathematics as Scientific Knowledge

Thomas Aquinas is in complete agreement with Aristotle as to the definition of science, its prerequisites, and the fact that mathematics itself is scientific knowledge. Thomas takes as his own the traditional definition of science, "certain knowledge through causes."[1] The following texts show this clearly:

> [S]cience is knowledge of a thing through its proper causes.[2]

and

> For science certitude of knowledge is required.[3]

In his Commentary on the Posterior Analytics,[4] St. Thomas, following the Stagirite, distinguishes science from what he calls understanding and from what he calls opinion. Science, he says, differs from understanding because it includes discourse from principles to conclusions. It is caused knowledge, i.e. knowledge of the conclusion is caused by knowledge of the principles upon which it is based,[5] and in the highest sense it is knowledge of the real causes of that which is known scientifically.[6] Science differs from opinion insofar as it is necessary or certain knowledge; it reaches conclusions through necessary principles.

In commenting upon Aristotle's famous definition of science with which he agrees (given in Part I, Chapter IV, p. 118), Aquinas points out that,

> To know scientifically is to know something perfectly, which is to apprehend its truth perfectly.[7]

Now to know something perfectly is to grasp it in its completed being (per-factum), to grasp it in the way that it has being. If it has being through a cause, then to know it perfectly demands knowing its cause.[8] Thomas also says to know scientifically is to apprehend the truth of something perfectly. Now truth refers to being as knowable; hence, for St. Thomas, the principles of being and of truth are the same.[9] To know a being truly is to know it as it has its being, for "everything is knowable in as far as it has being."[10] If a thing's being is from another, then to apprehend "its truth perfectly" is to know it as it really is, viz. as from that other, in other words as caused. And this, of course, is precisely what scientific knowledge does.

Since scientific knowledge is mediate knowledge based on principles, Thomas, following Aristotle, goes on to describe the requirements for the premises of demonstration, pointing out that these requirements follow from the very definition of demonstration in terms of its final cause, as a syllogism designed to produce scientific knowledge.[11] Aquinas discusses each of the requirements the Stagirite sets forth, viz. that the premises of a demonstration must be true, primary, immediate, prior to, better known than, and causes of the conclusion.[12] And he also explains that the premises

must involve universality, perseity, and convertibility.[13] It seems unnecessary to go through his discussion of these requirements, for it parallels so closely that of Aristotle which was given in Part I. In fact, I can find no significant difference between the two philosophers, though Aquinas is more detailed. (The reader can refer back to Part I, Chapter IV, pp. 119 ff., if he desires detail on these points.)[14]

We should perhaps recall, however, one point made in Part I concerning the premises of a demonstration, viz. that if one of them contains the definition of the subject the demonstration based thereon will be of the most rigorous, most complete type, and each premise itself will necessarily fulfill all of the aforementioned requirements. To repeat very briefly the analysis given in Part I: the most complete way that a predicate can be essentially related to a subject (mode of perseity) is when it is the very essence of the subject, i.e. when it is the definition. Furthermore, that which is the most basic commensurately universal intrinsic cause of the inherence of an attribute in a subject (modes of universality and convertibility) is the very essence of that subject. Therefore, that demonstration which has the definition of the subject of the syllogism as the middle term will be one which contains the most basic commensurately universal cause of the inherence of an attribute in a subject and one whose premises embody the most complete essential relation between their terms.

Moreover, if the premises express the definition, the syllogism will obviously contain the cause of the conclusion, viz. the essence, which accounts for the inherence of the predicate in the subject. The premises will also be primary and immediate since there is no intrinsic cause prior to the essence which would be its cause; they will be prior to the conclusion since the causes are prior to the effects; and they will be better known than the conclusion since what a thing is must be better known (in se at least) than that which follows from it.

In this connection, we noted that the science of mathematics according to Aristotle (Part I, Chapter IV, p. 122) is founded on just such basic premises, viz. that it contains the definitions of its subjects in its premises as the middle term and proceeds to demonstrate the necessary connection of properties with these subjects. Thomas Aquinas says exactly the same thing; the sciences of mathematics are based on definitions.

> In these sciences [viz. the mathematical]. . . we do not demonstrate something about one thing through another thing, but through the proper definition of that thing.[15]

Since this is so, clearly then, mathematical demonstrations are ultimately the most rigorous, most complete type and their premises fulfill the necessary requirements. (I say ultimately because, of course, the conclusions of one demonstration can become the premises of another, etc., etc., and in this way a science is enlarged. Nevertheless, the most basic premises of the demonstrations of mathematics contain

for Thomas the definitions of the objects under consideration.)

Since mathematical demonstrations are based on essential definitions, and these definitions (and essences) include matter according to St. Thomas (as we showed in Chapter II of this Part), it must follow that there can be mathematical demonstrations through both the intrinsic material cause of the essence and through its form. Now Thomas does occasionally speak as if all demonstrations in mathematics are demonstrations through form alone.[16] However, in at least two separate places, he refers to mathematical demonstrations through material cause,[17] and in his commentary on Aristotle's mathematical proof found in the Posterior Analytics, he explains that "whenever in mathematics something is demonstrated of the whole through the parts, there is seen to be a demonstration through the material cause, for the parts are to the whole according to the reason of matter. . . ."[18] He gives as an example of this type of demonstration the same proof cited by Aristotle, viz. the proof that every angle falling in a semicircle is a right angle. This demonstration uses as its middle term the parts of a right angle, viz. two half-right angles.[19] (Incidentally, it is interesting to note that Thomas even gives a reason why the Stagirite did not actually call this proof a demonstration through material cause. He says that "matter is more properly said of sensible things, [and] because of this he [Aristotle] did not wish to call it a material

cause. . . ."[20] We suggested a different reason in Part I, Chapter II, p. 44, viz. that Aristotle is undecided whether or not the mathematical essence contains matter.) The other type of mathematical demonstration, viz. through the form rather than the matter of the essence, would apparently occur when something is demonstrated of the parts through the whole itself. Actually, Thomas gives no explicit example of a mathematical demonstration of this type, but it would in general seem to be a proof based not primarily on the parts that make up a certain quantity but primarily on that which forms these parts into a certain whole, viz. their form, in the case of figures their shape. An example of this might be a demonstration of the properties of some parts of a circle, e.g. its chords or arcs, based upon the nature of a circle as a figure shaped (formed) so that it is everywhere equidistant from a common center point. Of course, in speaking about demonstrations through formal cause or demonstrations through material cause in mathematics, we must bear in mind what we have seen earlier, that, according to Aquinas, mathematical forms can never be totally separated from matter — even in thought — no more than the substantial form of a physical thing can be separated from its corresponding matter.[21] Hence, no term in a syllogism dealing with physical or mathematical entities could represent solely their matter or solely their form; every demonstration through either of these causes would have to contain both, i.e. the whole essence given in the definition. The difference between the two types of proofs

would apparently be that in the case of demonstration through material cause the emphasis would be placed on the material part of the essence (in the case of mathematics, the parts of the whole) while in demonstration through formal cause, it would be placed on the formal part of the essence (the form of the whole).[22]

We have said that since mathematical demonstrations are ultimately based on the definition of the mathematical essence, they are demonstrations in the most rigorous, most complete way and their premises necessarily fulfill the requirements for demonstration. The one requirement which needs special discussion is that the premises of a demonstration must be true. In Part I, a lengthy analysis was undertaken in order to determine what is necessary for the truth of the premises of mathematical demonstration according to Aristotle, and this question was related to the question of the freedom of the mathematician. In this connection, in the previous chapter, some questions have already been raised as to the freedom allowed the mathematician by Aquinas. We noted there that for St. Thomas, as for Aristotle, a proposition is true if it accurately states what is, and therefore that mathematical propositions will be true if they accurately state the way mathematicals are. Hence, since individual mathematicals as such are according to Thomas only in the imagination, since the judgments of mathematics, he says, terminate in the imagination, mathematical propositions to be true must conform to imagined mathematicals. But, if the mathematician must only conform to imagined entities, does this mean that

Thomas feels that this scientist is free to create and define any non-self-contradictory mathematical he pleases, and then proceed on the basis of his definitions to demonstrate the properties of the object he has arbitrarily created? We should recall, too, on this same point, that Thomas has said that the proximate foundation of mathematics is the intellect itself, not physical things. But then, is there any limitation to the mathematician's definitions, to creations he makes in imagination under the direction of the intellect? Is every premise, every definition, proposed by the mathematician true simply because he makes it so? All of these, of course, are basically the same questions we asked of Aristotle in Part I and Thomas's answer, when he gives one, is in many respects the same as the Stagirite's as we shall see.

The Truth of the Premises of Mathematical Science, or, the Freedom of the Mathematician

Aquinas is very explicit, as we saw in the previous chapter, that the mathematician must demonstrate the existence of mathematical entities by constructing them. Recall that the Stagirite too insisted that one must demonstrate that a certain quantity exists before he can define it or demonstrate its properties. Yet he never said exactly how the demonstrations of existence take place, though judging from the practice of his time it would have been by construction. (See Part I, Chapter IV, pp. 128-129.) Thomas leaves no doubt in the matter. Demonstrations of existence in math-

ematics are operational for, he says, they are by construction.[23] The following passage is part of his commentary on the Stagirite's statement that it is necessary to demonstrate the existence of certain mathematical objects. It also sets forth clearly Thomas's conception of the general procedure of the mathematician in his science.

> [T]here is supposed in these sciences [viz. the mathematical] those things which are first in the genus of quantity such as unity and line and surface and other such. These being presupposed, certain other things are sought by demonstration, such as the equilateral triangle, the square in geometry and other such things. These demonstrations are said to be, so to speak, operational, as is: On a given straight line to construct [constituere] an equilateral triangle. This having been proved, certain further passions are proved, as that its angles are equal or some other such thing. . . .[24]
>
> [underlining mine]

In addition to affirming the "operational" character of the mathematical demonstration of existence, this passage also clearly points out that the most basic mathematical entities, such as unity, line and surface, are not demonstrated but are "supposed." Certainly the mathematician cannot demonstrate the existence of every quantity with which he deals; this would involve him in an infinite regress. Indeed, Thomas affirms Aristotle's position that no science demonstrates the existence of the most basic elements with which it deals.[25] The mathematician, he says, supposes that those entities "which are first in the genus quantity" exist[26] (and by exist we mean exist in imagination since it is only there that individual mathematicals as such exist), and using these entities he goes on to construct, to demonstrate

operationally, certain figures or numbers (of which he has up till now only nominal definitions) composed of them. These constructions show that these composite entities do exist,

> [T]he geometer takes what this name "triangle" signifies and demonstrates that it is, e.g. when he demonstrates that on a given straight line an equilateral triangle is constructed. [constituere] [27]

Having done this, he then proceeds to prove the properties of these figures or numbers.

From the passage on page 275, it seems clear that like Aristotle, St. Thomas holds that it is only the most basic quantities, the ones supposed, which are directly abstracted from physical things. Other passages also lead to this conclusion. For example, he asserts that the principles of abstract mathematical quantities are known by abstraction from sensible particulars.[28] Now what does he mean by principles? Certainly, such propositions, such axioms, as the whole is equal to the sum of its parts, etc. But is that all? The most basic elements, the units, lines and surfaces, are also principles, principles from which come all the complex mathematical objects, figures and numbers, which the mathematician constructs. Therefore, Aquinas's statement that the principles of mathematics are known by abstraction could certainly be interpreted to refer not only to the most basic propositions of his science but also to the most basic quantitative elements.

We should now relate these two notions regarding the existence of mathematicals — some constructed, some, viz.

the most basic elements, assumed — to the questions raised above concerning the truth of the premises of the science of mathematics and the corresponding freedom of the mathematician. It seems clear that Thomas's position is that the premises will be true if they are definitions of constructible mathematical objects, for a real definition can only be given of something that exists or can exist.[29] As for the freedom of the mathematician, it would seem that he is perfectly free to construct any mathematicals he can, and this would apparently mean any quantities whose existence is not self-contradictory, and using their definitions as premises proceed to investigate their properties. But what about the most basic quantities which are the elements making up the constructed objects, those whose existence is supposed and which are apparently gained by abstraction? Is there any limitation on the mathematician here? Is any definition of such elements true and able to function as a basic premise in his science? Is he free to abstract any basic quantities, define them, and then go on and use them to construct other quantities? Or is there some limitation on what he can abstract and hence define? Certainly, the basic quantities St. Thomas refers to — units, points, lines, and surfaces — are most general. As he says, they are the ultimate, the first, in the genus of quantity. There is no built-in limitation with these as to what the mathematician can study for they are able to make up any mathematical object. It would appear that the most basic premises of mathematical science,

for Aquinas, contain the definitions of these absolutely first quantities, and that these definitions are true and these mathematical quantities exist simply because the mathematician says so, or better, because he simply "supposes" their existence once he has abstracted them. Indeed, as the most basic quantities in his science, how could the mathematician do anything else but "suppose" them? (We might point out here that it is not up to the mathematician as such to investigate the real foundation of these elements whose existence he assumes. He simply takes them and goes to work from there. It would seem to be the province of the philosopher of nature to show the basis in reality of these quantitative elements and hence to show that they are not mere mental fictions but actually real quantities considered, however, apart from their real existence.)[30] Finally, to repeat what we said above, since these are such basic elements and can make up any number (in the case of units), plane figure (in the case of lines), or solid figure (in the case of surfaces), they seem to contain no built-in limitation on the mathematician's freedom to construct any object he pleases, no more limitation than in saying that he must deal with quantity.[31]

And yet we must recall that Aristotle, who also referred to these general basic quantities as the elements of all mathematical objects, as a matter of fact seemed to limit the objects which could be constructed from them to quantities (both in the case of figures and numbers) which

corresponded to physical quantities. (See above Part I, Chapter IV, pp. 130-132.) Now it is also a matter of fact that St. Thomas too refers only to mathematical quantities which resemble physical quantities. The only geometrical figures and solids he mentions are those of Euclidean geometry. He too refers only to real numbers (not negative or imaginary), and he refers to them in terms which show that they are still being viewed as related to physical magnitudes. For example, he refers to numbers as surfaces, as solids, as two and three-dimensional, as squares, cubes, etc. (though he clearly recognizes such words are used metaphorically),[32] and he never refers to a number raised to any power higher than three, the cube.

It is true, of course, that by the thirteenth century mathematical objects were not considered to be simply idealized representations of actual physical quantities, at least not so much as they were in Aristotle's day. For one thing, the algebra had been introduced by the Arabs and put into Latin by some of the earliest translators.[33] Probably the most prominent mathematics book in Latin during Aquinas's time was the Liber Abaci by Leonardo Fibonacci (Leonardo of Pisa) published in 1202, and it was devoted to arithmetic and elementary algebra. Though it contained no recognition of negative or imaginary numbers,[34] it did have, in addition to the algebra, the use of the zero and of fractions and operations upon them.[35] Furthermore, during Aquinas's day symbols were more and more being used to represent quantities; in fact, one who pioneered this was a monk, Jordanus

Nemorarius, who in 1222 became general of the Dominican order.[36] Certainly, the use of symbols, instead of figures or numbers related to figures, to stand for quantities, indicates a view of mathematics which sees its objects removed from a direct one to one correspondence with physical quantities. In fact, the use of the zero alone indicates this for it has no physical counterpart and, indeed, for this reason it was looked upon by many as suspect.[37]

It is difficult to believe that Thomas Aquinas, who in other areas was so keenly cognizant of the newly introduced knowledge of his time, would not at least have been aware of these developments in the mathematics of his day. In fact, one author speculates that St. Thomas as a student used in his studies the Liber Abaci, for it was a commonly used text in the quadrivium.[38] Be that as it may, I have found no place in Aquinas where he explicitly refers either to the algebra or to the zero or to the use of symbolism in mathematics. If he was aware of these matters, he never said so.[39] He refers only to figures and numbers which correspond to physical quantities.

In this connection, recalling that Aristotle in a few places did refer to a general mathematical science which he called "universal mathematics," a science which seems to be akin to algebra (Part I, Chapter IV, pp. 135-136), one could ask what Thomas has to say about such a science. The answer is — practically nothing! For one thing, the two places in which the Stagirite actually uses the expression "universal mathematics" are in Aquinas's Latin translation

rendered as "universal science," which, of course, Thomas takes to mean metaphysics.[40] Three of the five other texts of Aristotle in which the Stagirite refers either to demonstrations of mathematics dealing with quantity in general, or simply to quantity in general, are in the latter books of his Metaphysics not available in translation to Aquinas.[41] Only in two places, in his Commentary on the Posterior Analytics, in discussing passages in which Aristotle states that commensurability is now proved of lines, numbers, solids, etc., not separately but by means of the common genus, quantity, in which they share — does Aquinas refer to propositions proved of quantity as such.[42] And he does no more than repeat what Aristotle says, very obliquely at that. Thus, except for these two oblique references, St. Thomas has no mention of a universal mathematical science akin to the algebra of his day. In this matter, then, one would have to say that Thomas has less to offer than the Stagirite.

Thus, it is clear that in any consideration of the freedom of the mathematician according to Thomas Aquinas, we must always keep in mind that he, like Aristotle, not only has no explicit reference to such freedom, he also, again like the Stagirite, refers only to those mathematical quantities which resemble physical quantities. And yet, unlike Aristotle, Thomas is living at a time when men are becoming more aware of the lack of any direct correspondence between mathematical and physical quantities; however, he makes no reference to these developments. Still, in spite of all

this, it seems undeniable that Aquinas's doctrine allows more freedom to the mathematician than does the Stagirite's. St. Thomas's teachings on the role of imagination, on the fact that the proximate foundation of mathematical objects is the intellect itself, on the use of construction to demonstrate the existence of mathematicals, and especially on the fact that mathematical judgments terminate not in the real world but in the imagination are at best only implied by Aristotle. All of these doctrines serve to liberate the mathematician more and more from any requirement to deal with quantities which match real quantities. There is nothing in these teachings which forces the mathematical scientist to limit himself to quantities resembling physical quantities. As we said earlier, Aquinas's thought would seem to be that the mathematician, after directly abstracting the most general and most basic elements of the genus quantity (points, units, lines, etc.) and defining them, is free to use these elements to construct in his imagination any objects he can, any objects in other words which are not self-contradictory, and then proceed to demonstrate their properties and relations. In terms of the truth of mathematical premises, as we have said, the premises of this science are true when they are the definitions of any constructible objects (or, of course, when they are the definitions of those most basic elements of quantity gained by abstraction and simply presupposed by the mathematician).

The Intermediate Sciences

All of the foregoing remarks on freedom in mathematics refer to the science of abstracted quantity, to what Thomas calls areas of "pure " mathematics. But there are for Aquinas other sciences concerned with quantity which do not treat simply of abstracted quantity, i.e. of real quantity not considered qua real, but rather of abstracted quantity considered qua real. I am referring, of course, to what Aristotle called the "more physical" mathematical sciences, such as harmonics, optics, mechanics, and astronomy. Thomas follows the Stagirite in affirming that these latter sciences differ from the purely mathematical ones inasmuch as they take abstracted quantity and apply it to sensible things. Thomas says,

> [S]uch sciences are quite different from those which are purely mathematical as geometry and arithmetic, because geometry does consider the line which has existence in sensible matter, which is the natural line, but it does not consider it insofar as it is in sensible matter according to which it is natural, rather it considers it abstractly, as was said. But perspective, on the other hand, takes the abstracted line as it is in mathematical consideration and applies it to sensible matter. Thus, the latter treats it insofar as it is physical but not insofar as it is mathematical.[43]

What he says of optics, perspective (which studies light mathematically),[44] he extends to harmonics (mathematical study of sound),[45] to astronomy (mathematical study of heavenly bodies, their distances, shapes, and motions),[46] and to mechanics (mathematical study of terrestrial bodies in motion acted on by other bodies). Each of these sciences is subalternated to a purely mathematical science — optics to geometry, harmonics to arithmetic, astronomy and probably

mechanics to both.[47] That is to say, they attain causal knowledge of what they study ultimately from the higher purely mathematical sciences.[48] Formally, then, these sciences are mathematical inasmuch as they demonstrate through mathematical, quantitative, causes. However, unlike Aristotle, Thomas does not designate them as mathematical sciences, or even as more physical mathematical sciences. It is true that materially they are physical for they deal with physical things, or more precisely with physical things in their quantitative aspects or aspects able to be treated quantitatively.

> [I]n their procedure what is physical functions as something material, while what is mathematical functions as something formal.[49]

Yet this does not make them simply physical sciences either for Aquinas. Instead he calls them intermediate sciences, <u>scientiae mediae</u>, not simply mathematical, not simply physical, but in between the two. In some places he does stress their mathematical side, the fact that they are scientific causal knowledge precisely because they are <u>mathematical</u>. And in these places he will say they are more like mathematical sciences.[50] But in other places, he will stress that which they study, physical things <u>qua</u> quantitative. And in these places he will say that in relation to their <u>term</u> they are more physical than mathematical.[51] (Obviously, they can and are never said to be physical sciences pure and simple since they do not study physical things <u>qua</u> physical, in terms of physical principles such as matter and form, but

qua quantity.)[52] Actually, rather than emphasize either their formal or their material part, it seems best to consider them as a unit and to say they are sciences in between the physical and the mathematical — in other words, to call them scientiae mediae. And this St. Thomas does in most places. One very clear text is the following:

> [T]here are three orders of sciences concerning natural and mathematical entities. Some are purely natural and treat of the properties of natural things as such, like physics, agriculture, and the like. Others are purely mathematical and treat of quantities absolutely, as geometry treats of magnitude and arithmetic of number. Still others are intermediate and these apply mathematical principles to natural things; for instance, music, astronomy, and the like.[53]

We have been saying that these intermediate sciences according to St. Thomas treat physical things qua quantitative. Let us become more exact about this. Recalling from Chapter I of this Part that quantity for Aquinas refers to either magnitudes or pluralities which are divisible into quantitative units, it follows that these intermediate sciences inasmuch as they consider quantitative features of physical things must consider either magnitudes or pluralities or both. For example, optics and astronomy, he says, apply geometrical principles to physical entities. But since geometry studies magnitudes, this must mean that optics and astronomy study physical magnitudes. However, optics treats of light and of color (e.g. the rainbow). Astronomy, while treating magnitudes (e.g. sizes of heavenly bodies and distances between them), also considers motion and time. In what sense are light and color (which are qualities), and mo-

tion and time magnitudes? They must be in order to have geometrical principles applied to them. We asked the same question of Aristotle in Chapter IV of Part I and Thomas's answer, like the Stagirite's, lies in the notion of per accidens or virtual quantity discussed in Chapter I of this Part.[54] Following Aristotle, Thomas says that time and motion are accidentally quantitative "because they are divided quantitatively as a result of the division of some quantity."[55] While in themselves, i.e. per se, motion and time are not quantitative magnitudes, they are quantitative per accidens, as a result of the quantitative magnitude "to which they belong."

> For time is divisible and continuous because of motion, and motion is divisible because of magnitude. . . the magnitude of the space through which it is moved.[56]

The same holds true of light and color. They too are per accidens quantitative magnitudes inasmuch as the subject to which they belong, air or some particular body, is quantitative.[57] Hence these physical entities are all per accidens magnitudes and therefore can have geometrical principles and notions, concerned with lines, angles, circles, spheres, etc., applied to them.

On the other hand, qualities such as sound, studied by the intermediate science of harmonics which applies arithmetical principles, must be discrete quantities and numerable. Now in order to assign numbers to various sounds some measure must be used. Measure, Thomas says, following Aristotle, is that whereby quantity is known,[58] and he points

out that qualities considered absolutely have no element (ratio) of quantity (After all, they are per se qualities.), and hence are not measurable nor able to be translated into numbers.[59] He adds, however, that considered relatively, i.e. in comparison with another quality of the same kind, they have an element (ratio) of quantity (In other words, they are virtual quantities.), and hence of measure, and so can be expressed numerically.[60] This is exactly what is done in harmonics; for example, a certain note of music is taken as a standard and other notes are then measured by some suitable instrument and expressed numerically as so many multiples of the standard. The various numerical ratios of one note to another are then investigated.

Of course, this view of St. Thomas's (and of Aristotle before him) that sound is virtually quantitative and so able to be numbered and treated by mathematical principles can be extended to many other qualities which are virtually quantitative, e.g. heat and heaviness (which he himself mentions),[61] and even to accidents other than qualities which can be measured. Aquinas points out that the effects and the causes of virtual quantity can be measured[62] and so treated by sciences of quantity — as, for example, today we measure heat by its effect on mercury in a glass tube or motion by the force which causes it. Certainly, actions and passions would be measurable in this way.

The point of this discussion is to show that for St. Thomas, as for the Stagirite, the intermediate sciences are

viewed quite broadly. That is, they study not only the obviously quantitative features of things but also nonquantitative features, features however which are per accidens or virtual quantities. In fact, according to Aquinas's principles, it would seem that the intermediate sciences are practically unlimited in the extent to which they can study the physical world for they can treat any features which are measurable and thus expressible quantitatively and able to be treated mathematically.[63] Thus, Thomas's notions of virtual or per accidens quantity provide the philosophical justification for the extension of the intermediate sciences that has taken place in modern times.

As for Aristotle, he too speaks of qualities as measurable by relation to other qualities,[64] and, of course, the notion of per accidens quantity is originally his.[65] Nevertheless, his treatment of these matters is somewhat less thorough than Aquinas's, his notion of virtual quantity in particular much less developed (See Part I, Chapter I, pp. 6-7.). Thus, we must grant that while Aquinas has adopted the Stagirite's principles, their detailed development in this area is to some extent his own accomplishment. And in general, Thomas's more detailed treatment of the subject of virtual quantity and its measurability seems naturally to tend to give more scope to the intermediate sciences than Aristotle envisaged.[66]

A greater reason for Aquinas's giving increased scope to the intermediate sciences is the fact that he is

much less optimistic than the Stagirite concerning the possibilities of a detailed specific natural science.[67] To be sure, St. Thomas agrees with his predecessor that a study of sensible things in terms of their most fundamental physical principles, matter and form, is more basic than a study of such things in terms of accidents of quantity, whether virtual or categorical.[68] He would agree, in other words, that in itself physics is a more basic study of physical things than are the intermediate sciences. However, Thomas also points out that, as a matter of fact, we do not know specifically, but only generically, the nature of any, or almost any, physical thing. We must almost always use accidental features ultimately to designate specific distinctions between things.[69] What this means for physics, since science begins with definition, is that we can not have a developed detailed philosophical science of physical things in terms of their specific features, but only in terms of their generic ones. What this means for the intermediate sciences St. Thomas does not say. But would it not be in accordance with his principles for us to add that it is the intermediate sciences that can provide scientific knowledge of sensible things in terms of their specific features, at least of those features which can be treated quantitatively? The intermediate sciences, then, though not of ontological principles and for that reason not as basic as physics, would be very important as providing scientific knowledge of the specific features of physical things, a knowledge which physics itself

can only rarely obtain.[70] Since, as we noted above, Thomas says that we must almost always use accidental features to obtain specific knowledge of things, what better accidental features to use than those most communicable[71] and about which we can be most certain,[72] viz. the quantitative and the qualitative treated quantitatively? Thus, it seems that precisely because St. Thomas is less optimistic than Aristotle as to the ability of man to obtain scientific knowledge of things in their specific ontological principles he allows a more prominent place to the intermediate sciences as more exact scientific studies of nature in its specific details.

The Speculative-Practical Character of the Mathematical Sciences

Let us conclude this chapter on the science of mathematics by considering St. Thomas's position on the speculative and/or practical character of pure mathematics and the intermediate sciences. We saw in Part I, Chapter IV, page 148 ff., that Aristotle distinguished speculative from practical knowledge on the basis of end and of principle. In many places, Aquinas repeats with approval this distinction:[73] speculative knowledge is for its own sake, its end is truth; practical knowledge is for the sake of making or doing, its end is either action or a product; the principles from which speculative science begins are located in the thing studied whereas in practical science they are in the scientist himself. (For a more detailed explanation of these points, see

Part I, Chapter IV, pages cited.)

However, if these distinctions are applied to pure mathematics, problems arise, the same problems which arose in reference to Aristotle. Thomas does agree with the Stagirite that in terms of end arithmetic and geometry are speculative for in both their goal is simply knowledge.[74] Yet from another point of view it is not so easy. For as we have noted, these sciences do involve a making, a product, inasmuch as they involve construction of their objects. Does this making make them practical also? Moreover, in terms of principles, the principles of this science are in a sense both within and outside the knower. Quantity, of course, really exists. Yet the abstracted quantity treated by mathematics exists as mathematical only in the mind, and furthermore the mathematician himself is the proximate source of the objects he constructs. Do these factors also make these sciences practical, at least partly?

We asked this same question of Aristotle. And we noted that he has practically nothing to say on the subject. St. Thomas, however, though to my knowledge he never refers to these sciences as practical, has much to offer. For one thing, though he does unquestionably consider these sciences to be speculative (They are one of the threefold division of the speculative sciences he inherits from Aristotle.),[75] at the same time he explicitly calls them arts, for he ranks them among the seven liberal arts.[76] And he says why he calls them arts:

> [T]hese are called arts among the other sciences because they not only involve knowledge but a certain work which is directly a product of reason itself.[77]

Such sciences, which are also arts, he distinguishes from sciences which are not arts:

> Other sciences, like divine and natural sciences . . . do not involve a work produced, but only knowledge, and so we cannot call them arts, because, according to the Metaphysics, art is called "productive reason."[78]

Actually, art in the strictest sense for St. Thomas involves a making which takes place in something external to the maker. That is the way he defines it in his Commentary on the Nicomachean Ethics.[79] However, there is another type of art which he says is art by a kind of comparison, a type which does not involve a transformation of external matter but instead involves a making which remains in the knower. These latter, arts by similitude, he calls liberal arts, the former, art in the strict sense, servile arts.[80] He explains this distinction clearly in his Summa.

> Even in speculative matters there is something by way of work: e.g. the making of a syllogism or of a fitting speech, or the work of counting or measuring. Hence whatever habits are ordained to such-like works of the speculative reason, are, by a kind of comparison, called arts indeed, but liberal arts, in order to distinguish them from those arts that are ordained to works done by the body, which arts are, in a fashion, servile. . . .[81]

Clearly then, the mathematical sciences of arithmetic and geometry, inasmuch as they involve a making, a constructing, of their objects, a making which remains in reason and so does not involve bodily actions which transform external things, are fittingly called liberal arts by St. Thomas.[82]

And yet, he insists that they are also speculative (even the most recently cited passage shows that). In terms of their end this is quite reasonable. The ultimate goal of these sciences is simply to know, to know all that is possible about magnitudes and numbers. Their end is truth — not action or making.[83] Making does take place but it is not the goal of the science; the making is only to provide objects whose properties can then be investigated. As one author puts it, in mathematics "it is not a question of knowing for the sake of making, it is a question of making for the sake of knowing."[84] The end of mathematics is not to construct quantities but to know the quantities which have been constructed.

However, from the point of view of the location of the principles of these sciences, their speculative character is not so evident. As we said in Part I, Chapter IV, in reference to the Stagirite's teachings, inasmuch as the complex mathematical objects are constructed, the principles, the source of such construction, of their very existence, reside in the knower. He makes certain figures, for example, by bisecting lines, drawing perpendiculars, etc., in accordance with some ideas and technical knowledge which he possesses. In fact, one of the first steps in mathematics consists in learning how to construct desired mathematical objects. On the other hand, once the objects are constructed the mathematician seeks to determine their properties based on the very nature of the objects themselves.[85] The respective es-

sences of the objects themselves become, then, the principles of his demonstrations. Thus, it seems that from the point of view of the location of their principles mathematics is partly practical, inasmuch as the objects are originally constructed according to principles in the knower, and partly speculative, inasmuch as the constructed objects are studied in terms of their intrinsic causal principles.

In the very beginning of his Summa St. Thomas gives different, though not totally different, bases to distinguish speculative and practical knowledge.[86] In that place he points out that some knowledge is speculative only, some practical only, and some both speculative and practical. He explains that knowledge is called speculative in three ways:

1. In relation to the things known: things which are not operable by the knower can be known only speculatively.
2. As regards the manner of knowing: if the knowledge is of operable things in a speculative manner, i.e. by considering such things in terms of their ultimate formal principles — "as for instance, if a builder were to consider a house by defining and dividing, and considering what belongs to it in general."[87]
3. As regards the end: the end of practical knowledge is operation; of speculative, truth.

And he concludes:

> Knowledge which is speculative by reason of the thing itself known [1] is merely speculative. But that

> which is speculative either in its mode [2] or as to its end [3] is partly speculative and partly practical.[88]

We have already seen that arithmetic and geometry are speculative according to St. Thomas in terms of end (#3). It remains to investigate them in terms of conditions #1 and #2. Certainly mathematical objects are operable, as we have seen more than once; they are constructed, generated, destroyed by us. Hence mathematics is not purely speculative (#1). As regards the manner of knowing (#2), it would seem that the same distinctions must be made that were made concerning the principles of the science. Certainly part of pure mathematics consists in studying how to construct various objects; a line, for example, is studied in terms of how it can be drawn perpendicular to another. But once the operation is finished, once the object is constructed, e.g. a right triangle, it is studied in terms of its intrinsic principles, its form and essence, and demonstration is sought of its properties. Thus in terms of mode, as in terms of principles, arithmetic and geometry are apparently partly speculative and partly practical for St. Thomas.

To conclude this discussion, it seems necessary to say that in accordance with St. Thomas's principles arithmetic and geometry are for him partly practical (and so he calls them arts) and partly speculative sciences. They deal with an operable object. In terms of mode they consider these operable objects partly as operable, partly simply as knowable; the principles of the science are located some in

the knower, viz. the principles of construction, some in the object; in terms of end their ultimate goal is truth not operation. By way of comparison with Aristotle's teaching on the subject, it should be said that Aquinas's discussion of the differences between speculative and practical knowledge is much more thorough than the Stagirite's. As far as I know, Aristotle never mentions a knowledge which is partly practical and partly speculative. He does, of course, consider the pure mathematical sciences to be speculative, but except for one rather ambiguous reference to the mathematical "arts"[89] (ambiguous because it is not at all clear to what mathematical sciences he is referring), the Stagirite makes no mention of their practical or artistic side.

What about the intermediate sciences — optics, harmonics, astronomy, and mechanics? These do not study objects constructed by the knower himself but natural objects already present before him. Are they speculative or practical or both for Aquinas? Certainly, the natural things which they treat are operable by the knower. Sound can be manipulated and is manipulated to make music; light can also be manipulated, as can physical forces and bodies, even the heavenly bodies (though Thomas may not have thought so).[90] Therefore, these sciences are not purely speculative in the first way cited above (#1). However, as regards the manner in which they study their operable objects (#2 above), it would seem that they are speculative since they do not seek to know so much how to manipulate objects as to know them in their ulti-

mate quantitative principles. For example, harmonics is not primarily a study of how to make music but a study of the ultimate quantitative principles involved in music; likewise for the other intermediate sciences. Also in terms of end (#3) the intermediate sciences have as their goal truth, viz. to know the quantitative properties of this or that material entity, rather than operation. Finally, as regards the principles of these sciences it would seem that in this matter they differ from the pure mathematical disciplines. The principles of the intermediate sciences are totally in what they study because they study not things the scientist constructs but things already given in the physical world.[91] For example, the principles upon which the science of astronomy is based are the ultimate quantitative properties, motions, distances, etc., really of and in heavenly bodies, not in the knower. So also for the other scientiae mediae. Thus, in conclusion, the intermediate sciences, like those called pure mathematics, are both speculative and practical: speculative in terms of mode, end and principles; practical inasmuch as they treat of operable objects.

In comparing Thomas's and Aristotle's views on this subject, we must recall that the former's distinction between speculative and practical is much more thorough than the latter's. Therefore, while Aristotle, on the basis of their end and principle, seemed to consider the intermediate sciences to be totally speculative (At least, he said nothing else except, again, for his ambiguous reference to mathematical

arts.), Thomas in line with his more complete distinctions would have to consider them to be both speculative and practical.

We will conclude this chapter as we have all the others by briefly comparing Thomas's and Aristotle's doctrines. The medieval saint is in agreement with the Greek philosopher on the general notion of science, and in maintaining that mathematics is a science in the highest sense because its demonstrations are based on the definitions of its objects. (We also find Thomas, like Aristotle, speaking of mathematical demonstration through both formal and material cause.) However, when we come to the question of the truth of these definitions and to the closely related question of the freedom of the mathematician in his choice of them, we find that St. Thomas has in many places completed the Stagirite's doctrine. For one thing Aquinas explicitly states (Aristotle never does.) that the existence of mathematical objects is demonstrated by their construction. Apparently for St. Thomas only the most basic elements of quantity are directly abstracted (This also seems to be the case for Aristotle.); the existence of the rest comes from their construction. But since construction is only of individuals and since Thomas clearly holds that individual mathematicals exist only in the imagination and hence that mathematical judgments can terminate only there (Again, Aristotle never explicitly makes either statement.), he thereby provides a strong basis for the freedom of the mathematician in creating the objects he would study. Using

the very general abstracted quantitative elements, he is apparently free to construct any non-self-contradictory quantitative object he pleases in his imagination. In terms of the premises on which his science is based, they need only conform to constructible objects or to the most basic abstracted elements (whose existence is assumed) in order to be true. And the abstracted elements themselves are so general that any non-self-contradictory quantity can be formed from them. It is true that we did say in Part I that in the Stagirite's teachings he has implicitly given the bases for saying that the mathematician has a good deal of freedom in his science. However, since Aquinas is so much more explicit on the abovementioned points, we not only find in his teachings implicitly given bases for such freedom but we find clear statements of these bases. And yet in this matter of freedom, it is true of Aquinas as it was of Aristotle that he never overtly mentions it. As far as the actual objects to which they refer are concerned, both philosophers seem to limit mathematical quantities to those resembling physical quantities.

As for what Aristotle called the "more physical" mathematical sciences, Thomas does have points of difference with his predecessor's doctrine. He agrees that these sciences do consider the quantitative features of physical entities, not quantity simply in abstraction. However, unlike the Stagirite, he does not call them simply mathematical sciences, though he agrees that formally they are such. Thomas

calls them instead intermediate sciences, *scientiae mediae*, since they are also materially physical. Furthermore, because of his more detailed treatment of virtual quantity and its measurability, he gives these sciences more scope in studying physical phenomena than was ever clearly envisaged by Aristotle. Even more important in giving added scope to these sciences is the fact that Aquinas is less optimistic than the Stagirite as to the possibility of a physical science of nature in its specific qualitative and intelligible features. The intermediate sciences, in other words, are invaluable in providing a more detailed scientific study of nature than is obtainable by physics.

Finally, Thomas is much more thorough than Aristotle in distinguishing speculative and practical knowledge. Therefore, the speculative-practical character of both pure mathematics (which he also calls liberal arts) and the intermediate sciences is more evident in his doctrine than in that of the Greek philosopher. The latter apparently viewed both types of science as completely speculative — though he does have one obscure reference to mathematical "arts."

FOOTNOTES

[1]This is not to say that he explicitly uses this phrase, no more than did Aristotle. But it is an accurate and concise formulation of the position of both men. Incidentally, as Defarrari, A Lexicon. . ., p. 998, points out there is a loose and a strict sense of the term scientia in Aquinas. In the loose sense, it means any intellectual knowledge certain and evident, the opposite of faith and opinion on the one hand and ignorance on the other; in the strict sense, it means certain knowledge through causes. It is the latter sense with which we are concerned in this chapter.

[2]C. G., I, 94, 3.

[3]De Veritate, 11, 1c.

[4]In I Posterior Analytics, L. 36 and 44.

[5]S. T., I, 14, 1, ob 2; In I Posterior Analytics, L. 4.

[6]C. G., I, 94, 3; In I Posterior Analytics, L. 42, 7-8.

[7]In I Posterior Analytics, L. 4, 5.

[8]Ibid.

"For the principles of the existence of a thing and of its truth are the same (as is evident in Metaphysics II). It is necessary therefore for the one who has scientific knowledge, if he knows perfectly, to know the cause of the thing known scientifically."

See Morrison's discussion of scientific as perfect knowledge, A Definition of Mathematics, pp. 131-34.

[9]In I Posterior Analytics, L. 4, 5.

[10]S. T., I, 16, 3c.

[11]In I Posterior Analytics, L. 4, 9.

[12]In I Posterior Analytics, L. 4.

[13]In I Posterior Analytics, L. 9, 10, and 11.

[14]An even more detailed analysis of the requirements for demonstration can be found in Owen Bennett, The Nature of Demonstrative Proof — according to the Principles of Aristotle and St. Thomas Aquinas (Washington, D.C., 1943), Chapter III. See also Morrison, A Definition of Mathematics, Chapter III; Simmons, The Scientific Art. . ., Chapter XIX.

[15]De Trinitate, VI, 1c; see also In I Posterior Analytics, L. 22, 11.

[16]One text is In I Posterior Analytics, L. 4, 16:

"[I]n mathematics, . . . because of the abstraction from matter, demonstrations are had from formal principles alone."

Another is in the De Trinitate, VI, 1c. In each passage, he apparently means form apart from sensible matter. At least this is the way he interprets Aristotle's statements that mathematics demonstrates only through form.

[17]De Trinitate, V, 3, ad 4:

"Mathematics does not abstract from every kind of matter but only from sensible matter. Now the parts of quantity which seem to be in a way the basis for a demonstration by means of a material cause are not sensible matter; rather, they pertain to intelligible matter. . . ."

See also In II Posterior Analytics, L. 9, 5. Those like Mahoney, The Notion of Quantity. . ., pp. 21-22, who state categorically that demonstration is only through formal cause in mathematics must simply ignore these texts.

[18]In II Posterior Analytics, L. 9, 5.

[19]Thomas's proof that every angle falling in a semicircle is a right angle is given both in In II Posterior Analytics, L. 9, 6 and In IX Metaphysics, L. 10, 1891-93.

[20]In II Posterior Analytics, L. 9, 5.

[21]De Trinitate, V, 3c.

[22]Thomas states explicitly that there are no demonstrations through final or efficient cause in mathematics:

"[F]rom the fact that there cannot be an end in the case of immobile things, it seems to follow that in the mathematical sciences, which abstract from matter and motion, nothing is proved by means of this cause. . . ."
(In III Metaphysics, L. 4, 375)

"Now, it is becoming that everything should have an

efficient cause in proportion to its being. And so, although the object of mathematics has an efficient cause, still, its relation to that cause is not the reason why it is brought under the consideration of the mathematician, who therefore does not demonstrate that object from its efficient cause." (S. T., I, 44, 1, ad 3)

However, as the second text indicates, mathematicals do have an efficient cause, ultimately the mathematician himself, but more proximately one mathematical may serve as the agent in the production of another. Of course, mathematicals also have a final cause; the mathematician has some end for which he makes and manipulates his quantities. Now, as we pointed out in reference to Aristotle (Part I, Chapter IV, footnote 22), these efficient and final causes do enter into some mathematical demonstrations, viz. "demonstrations" of existence. These demonstrations are ones which involve construction of figures or numbers (See the next section of Chapter IV for a discussion of these demonstrations.); certainly the efficient and final cause of a constructed figure are integral to its coming to be, to its demonstration of existence. Of course, once the existence of a particular mathematical quantity has been demonstrated, it is investigated in terms of only its essence, i.e. its intrinsic causes, material and formal. Once it exists then, none of its properties are demonstrated through its final or efficient causes. (The whole of footnote 22, Part I, Chapter IV, p. 156, has a somewhat more thorough treatment of this same point.)

[23]Maritain, The Degrees of Knowledge, in many places refers to construction in mathematics as showing An Est, pp. 55, 163, 181 footnote 1, 203.

[24]In I Posterior Analytics, L. 2, 5. Thomas also refers to construction in mathematics as the means of demonstrating the existence of mathematicals in In II Posterior Analytics, L. 6, 4.

[25]In I Posterior Analytics, L. 18, 9; L. 2, 2 and 3.

[26]In I Posterior Analytics, L. 18, 4:

"Just as in mathematics, one begins by accepting what unity is, which is the principle, and what straight line is, and what triangle is, which are not principles, but passions. But that unity is, and that magnitude is, the mathematician accepts as principles, but the others he demonstrates, i.e., those things which follow from the principles. For he demonstrates the equilateral triangle, and the right angle, and also that this line is straight."

[27]In II Posterior Analytics, L. 6, 4.

[28]In I Posterior Analytics, L. 30, 5:

"But the principles of abstract things, from which demonstrations proceed in these matters, are not manifested to us except from certain particulars which we perceive through the senses. Thus the universals from which demonstration proceeds are made known to us only by induction."

In VI Ethics, L. 7, 1209 uses the term abstraction in the same context. Certainly, induction is a type of abstraction. See also Greenwood, Études sur. . ., p. 85. Thomas also refers to these principles as arising from inductions based on particulars in the imagination. (In I Ethics, L. 11, 137) Principles in this context would have to mean propositions in the intellect — not quantitative elements. On this point, see Alvarez, "La filosofìa della matematica . . .," 149.

[29]In II Posterior Analytics, L. 7, 7; In I Posterior Analytics, L. 2, 5:

"[T]here are no definitions of nonbeings."

[30]If I understand him correctly, Maritain, The Degrees of Knowledge, holds this and, furthermore, states that it is Thomas Aquinas's position, p. 40, footnote 2, and p. 46.

[31]For some who have discussed the freedom of the mathematician in his choice of object, see Morrison, A Definition of Mathematics, pp. 43-4, 70; Maziarz, The Philosophy of Mathematics, p. 195; Maurer, "A Neglected Thomistic Text. . .," 192.

[32]In V Metaphysics, L. 14, 974; In V Metaphysics, L. 16, 989-991. Aristotle, of course, also recognized that the use of terms applying directly to continuous quantity to signify properties of discrete quantities is metaphorical.

[33]Maurer, "A Neglected Thomistic Text. . .," 185.

[34]First used by Raffael Bombelli, 1550. (D. Struik, A Concise History of Mathematics [New York, 1948], p. 114)

[35]For information on this book, its author, and the general state of mathematics in the thirteenth century, consult Cajori, A History of Mathematics, pp. 117-125; H. Eves, An Introduction to the History of Mathematics (New York, 1961), p. 209. See also Greenwood, Études sur. . ., pp. 66 ff.

[36]Eves, . . . History of Mathematics, p. 209.

[37]Cajori, A History of Mathematics, p. 121; Greenwood, Études sur. . ., p. 69.

[38]Greenwood, Études sur. . ., p. 65. However, Vernon

Bourke, in his most recent work, Aquinas' Search for Wisdom (Milwaukee, 1965), says that the quadrivium was no longer followed in the thirteenth century because masters proficient in the mathematical sciences were scarce, p. 22. And he gives nothing to support the view that Thomas was taught the "new mathematics."

[39]Maurer makes the same point, "A Neglected Thomistic Text. . .," 185.

[40]One of Aristotle's texts is in Metaphysics, VI, 1, 1026a 25-27:

"[N]ot even the mathematical sciences are all alike in this respect — geometry and astronomy deal with a certain particular kind of thing, while universal mathematics applies alike to all."

The translation St. Thomas has reads (In VI Metaphysics, L. 1):

"[N]ot even in the mathematical sciences is the method the same, because both geometry and astronomy deal with a particular kind of nature, whereas the first science is universally common to all."

His commentary on this is at 1169. The other text of Aristotle's is Metaphysics, XI, 7, 1064b 8-9; it reads:

"Each of the mathematical sciences deals with some one determinate class of things, but universal mathematics applies alike to all."

The translation St. Thomas has reads (In VI Metaphysics, L. 7):

"For each of the mathematical sciences deals with some one determinate class of things, but a universal science is common to all."

His commentary on this passage is at 2266-67.

[41]The passages I am referring to are in Aristotle's Metaphysics, XIII, 2, 1077a 9-10; XIII, 3, 1077b 17-20; XIV, 1, 1087b 15-18.

[42]Aristotle's texts are from the Posterior Analytics: I, 5, 74a 17-30 — Thomas's Commentary, I, L. 12, 8; II, 17, 99a 8-11 — Thomas's Commentary, II, L. 19, 3.

[43]In II Physics, L. 3, 164 (pp. 113-114 of Kocourek's translation).

[44]In I Posterior Analytics, L. 41, 3; In III Meta-

physics, L. 7, 412.

[45]De Trinitate, V, 3, ad 6; In III Metaphysics, L. 7, 412; In I Posterior Analytics, L. 41, 3.

[46]De Trinitate, V, 3, ad 5, ad 6; S. T., I-II, 54, 2c (Dominican trans.); In III Metaphysics, L. 7, 411; In I Metaphysics, L. 13, 202.

[47]In II Physics, L. 3, 164. See also In I Posterior Analytics, L. 15, 5; L. 41, 3.

Actually, I can find no place where Thomas says to what mathematical sciences mechanics is subordinated. Inasmuch as it, in general, studies on the terrestrial level what astronomy studies on the celestial, viz. bodies and their interactions, distances, and motion, I presume it is generally the same as astronomy, i.e. subordinate to arithmetic and geometry.

[48]See the whole of Thomas's detailed discussion of subordination, In I Posterior Analytics, L. 25. On subordination, see also the article by B. Mullahy, "Subalternation and Mathematical Physics," Laval Théologique et Philosophique II (1946) and Maritain, Philosophy of Nature, the section entitled "The Subordination of the Empiriological Realm to Mathematics or to the Philosophy of Nature," pp. 107-114.

[49]De Trinitate, V, 3, ad 6. He makes the same point in In I Posterior Analytics, L. 25, 2-4. See Maritain, The Degrees of Knowledge, p. 42, footnote 2.

[50]De Trinitate, V, 3, ad 6:

"These sciences have, however, a closer affinity to mathematics, because in their procedure what is physical functions as something material, while what is mathematical functions as something formal. . . . Thus they demonstrate their conclusions concerning natural things, but by means of mathematics."

See also In I Posterior Analytics, L. 25, 2-4.

[51]In II Physics, L. 3, 164:

"However, although sciences of this sort are intermediates between natural science and mathematics, they are here said by the Philosopher to be more natural than mathematical, because each thing is named and takes its species from its terminus. Hence, since the consideration of these sciences is terminated in natural matter, then even though they proceed by mathematical principles, they are more natural than mathematical sciences."

In IV Physics, L. 7, 477.

[52]In I Posterior Analytics, L. 41, 11.

[53]De Trinitate, V, 3, ad 6. Other places where he refers to these sciences as intermediate are In I Posterior Analytics, L. 41, 3; In II Physics, L. 3, 164; In I Metaphysics, L. 13, 202.

[54]I am not implying that per accidens and virtual quantity mean exactly the same for Aquinas. They definitely do not. (Cf. In I Sententiarum, D. XVII, Q. II, art. I, ad 1 [Parma, VI, p. 143]) Per accidens quantity refers to that entity which is not per se quantitative but which can be considered quantitatively because it is present in or related to per se quantity. (See the text referred to in the following footnote.) Virtual quantity, while it includes entities which are per accidens quantity, refers also to entities which can in no way be present in or related to per se quantity, for example, God. In other words, per accidens quantity seems to be a subdivision of virtual quantity.

[55]In V Metaphysics, L. 15, 985.

[56]In V Metaphysics, L. 15, 985. See also De Trinitate, V, 3, ad 5.

[57]In V Metaphysics, L. 15, 983-4.

[58]In X Metaphysics, L. 2, 1938.

[59]In X Metaphysics, L. 2, 1941-2.

[60]Ibid.

"In the case of dimensions no one doubted that they were quantities and that they were properly susceptible to measurement, but in the case of weight and of speed there could be a difficulty because these seem to be qualities rather than quantities. He [Aristotle] therefore explains how these pertain to the genus of quantity, and how they are susceptible to measurement. . . . For each of these terms [i.e. heaviness and rapidity] is used in two senses. In one sense the term heavy is used without qualification of anything that has an inclination to be borne towards the center of the earth, without taking into consideration how great its inclination is; and in this sense heavy does not refer to the category of quantity, and it is not susceptible to measurement. In the other sense it is used of one thing in comparison with something else, namely, of what exceeds something else in terms of the abovementioned inclination; for example, we say that earth is heavy in comparison with water, and that lead is heavy in comparison with wood. Therefore it is by reason of this excess that some notion of quantity and

measure is found. The term rapid is similarly used in two senses. In one sense it is used without qualification of anything that has any motion; and in a second sense it is used of anything that has an excessive motion. And in one respect the notions of quantity and measure properly apply to it; and in another respect they do not."

See Mark Heath's discussion of this, "Can Quantities Be Measured?", 50-55.

[61] In X Metaphysics, L. 2, 1942.

[62] S. T., I, 42, 1, ad 1. Mark Heath, "Can Quantities Be Measured?", 57.

[63] Many authors have stressed the wide scope of the intermediate sciences: Maritain, The Degrees of Knowledge, pp. 145-6; Maziarz, The Philosophy of Mathematics, p. 198; Collingwood, Philosophy of Nature, pp. 71-3.

[64] Metaphysics, X, 1, 1053 b5 and X, 2, 1053 b27. See Mark Heath, "Can Quantities Be Measured?", 34.

[65] Categories, 5b, 1 ff.; Metaphysics, V, 12, 1020a 4 ff.

[66] Owens points out how Aristotle seemed to have no inkling of the possible range of the more physical mathematical sciences, "The Aristotelian Conception. . .," 214-5.

[67] Owens, "The Aristotelian Conception. . .," 213-4, refers to the naivete of Aristotle in this regard and contrasts Thomas's more moderate view with the Stagirite's. See also "Our Knowledge of Nature," Part II.

[68] De Trinitate, V, 3, ad 2:

"For the mathematician does not consider the body which is found in the category of substance, according as its parts are matter and form, but he rather considers body according as it is in the category of quantity constituted by three dimensions; and body in this sense is related to body in the category of substance, of which physical matter is a part, as an accident to its subject."

S. T., I, 7, 3c; In I De Caelo, L. 1, 7 (Parma, XIX, p. 3).

[69] In some texts Thomas asserts categorically that the essential principles of things are unknown. In others, he says they are "almost impossible" to know. (Cf. In I De Anima, L. 15; In II De Anima, L. 2, 237; C. G., I, 3; De Veritate, 10, 1, etc. Maritain, The Degrees of Knowledge, pp. 207-8, discusses this point and gives many other texts in footnote 2 on these pages. See also the article by R.

Gehring, "The knowledge of material essences according to St. Thomas Aquinas," Modern Schoolman, XXXIII (1956), and Owens, "Our Knowledge of Nature," 73-76.

[70]Maritain emphasizes the role of the mathematical science of nature in providing detail not obtainable by physics, the philosophy of nature. See his section, "The Philosophy of Nature and the Natural Sciences are Mutually Complementary," pp. 93-8, Philosophy of Nature. Vincent Smith, The General Science. . ., makes the same point, pp. 154-5.

[71]Collingwood, Philosophy of Nature, pp. 67-8.

[72]In II Metaphysics, L. 5, 336; In II De Anima, L. 2, 245; Greenwood, Etudes sur. . ., pp. 80-4.

[73]In XI Metaphysics, L. 7, 2253-55; In VI Metaphysics, L. 1, 1152-55; In XI Metaphysics, L. 7, 2265. In I Posterior Analytics, L. 41, 7; De Trinitate, V, 1c.

[74]In I Posterior Analytics, L. 41, 7; In VI Metaphysics, L. 1, 1160; In I Ethics, L. 11, 136; De Trinitate, V, 1c.

[75]He refers to this division in De Trinitate, V, 1c; see also In VI Metaphysics, L. 1, 1160.

[76]De Trinitate, V, 1, ad 3; see Maurer's footnote on pp. 10-11 of his translation of this work.

[77]In I Metaphysics, L. 1, 33.

[78]Ibid.

[79]In VII Ethics, L. 3, 1154-5.

[80]On this discussion, cf. Mullaney, "The Liberal Arts. . .," 488-9 and Simmons, The Scientific Art. . ., p. 316.

[81]S. T., I-II, 57, 3, ad 3 (Dominican trans.). He makes the same distinction in the De Trinitate, V, 1, ad 3.

[82]Robert Smith, "The Liberal Arts," 83; Maurer, "A Neglected Thomistic Text. . .," 190-1, footnote; B. Mullahy, "The Nature of the Liberal Arts," New Scholasticism, XXIII (1949), 376.

[83]De Trinitate, V, 1c; In I Ethics, L. 11, 136.

[84]Mullahy, ". . . Liberal Arts," 491.

[85]Smith, "The Liberal Arts," 81.

[86]S. T., I, 14, 16c; J. Oesterle, "Theoretical and Practical Knowledge," Thomist, XXI (1958), 161.

[87]Ibid.

[88]Ibid.

[89]Metaphysics, I, 1, 981b 23-24.

[90]Today the possibility of mining the heavenly bodies is not far fetched.

[91]I realize that Thomas does call the entire quadrivium liberal arts, and this includes harmonics (music) and astronomy (De Trinitate, V, 1, ad 3). See also Robert Smith, "The Liberal Arts," 83. However, I cannot see that there is really any art, any making, in the intermediate sciences. It is true that in the medieval schools what was called harmonics (music) did include the making of songs. But I would distinguish this art of music-making from the intermediate science of harmonics which simply investigates the quantitative properties of music. (Mullahy, ". . . Liberal Arts," 383, has this distinction.) Perhaps in the quadrivium the two were simply lumped together and this is why Thomas calls harmonics (music) a liberal art.

As for astronomy, it seems to be called an art because it (and indeed all the intermediate sciences) involves physical operations of measuring and operations of numbering. (See the text on page 292, cited by footnote 81.) However, I really cannot see how measuring and numbering involve a real making. One might just as well say that metaphysics is an art because in it we make judgments, or natural science because in it we make classifications. Perhaps, at Aquinas's time, the astronomy of the quadrivium also included the study of how to make physical measurements. If so, in this sense, it could be called an art.

CHAPTER V

CONCLUSION

Throughout the foregoing chapters of Part II and especially at the end of each one, a comparison has been made between the doctrine of Aquinas and that of Aristotle on the points treated in that particular chapter. It seems unnecessary for me to repeat that detail here. Instead, I will attempt to list in rather succinct fashion what seem to me to be the most significant areas of agreement and difference between the two philosophers and refer the interested reader to the more complete treatment of these points in the proper chapter.

First, the general areas of <u>explicit</u> agreement:

1. Thomas follows Aristotle in holding that the general object of mathematics is quantity which is an accident and he accepts the Stagirite's definition of this accident. (Part II, Chapter I)
2. Thomas agrees with the Greek philosopher on the peculiar nature of the quantities treated in mathematics; he agrees that they are gained by abstraction from quantified physical things; he accepts Aristotle's notion of abstraction as

a mental subtraction of all nonquantitative features from physical things. In other words, he agrees that the mathematician treats real quantities but not qua real. Following from this, Thomas agrees that mathematicals as such exist only in the thought of the mathematician. (Part II, Chapter II)

3. Thomas accepts Aristotle's view that intelligible matter is included in the objects of mathematics as their principle of individuation and that the accident of quantity must include in its intelligibility a subject of the quantity. He also follows the Stagirite in designating the parts of mathematical quantities as their intelligible matter. (Part II, Chapter II)
4. Thomas accepts the notion of science offered by the Stagirite and agrees that mathematics is a science in the highest sense since it uses definitions as its middle terms. He also agrees that mathematics is ultimately founded on statements which assert both the existence and the definition of the most basic quantitative elements of this science. He agrees further that these basic elements are gained by abstraction and that the existence of all other objects in this science must be demonstrated. (Part II, Chapter IV)
5. Thomas accepts the Stagirite's view concerning

the possibility of sciences which treat physical things qua quantified, sciences which are scientific precisely inasmuch as they apply mathematical principles. He also follows Aristotle in extending these sciences so that they are not limited to treating only the per se quantified but study also the per accidens quantified and, in general, seem to be able to treat any features of physical things which are measurable and thus able to be transferred into quantitative terms. (Part II, Chapter IV)

Second, I will list the areas of difference. I say difference rather than disagreement because in most cases it is not that Thomas disagreed with Aristotle in doctrine, but that he developed or made explicit what the Stagirite only implied.

1. For example, St. Thomas has much more detail on the distinction, which is present in Aristotle, between virtual and categorical quantity. He explicitly states what the Stagirite implies, that pure mathematics deals only with categorical quantity. (Part II, Chapter I)
2. Thomas also states clearly the close relation of quantity and matter and the ontological priority of quantity to all other accidents; Aristotle explicitly mentions neither of these points though both are implied in his doctrine. (Part II,

Chapter I)

3. Aquinas has more of a development of the notion of mathematical quantities as in a way *entia rationis*. He stresses that their proximate foundation is in the intellect itself. (Part II, Chapter II)
4. St. Thomas explicitly states that the demonstration of the existence of mathematical objects takes place by constructing them; this can only be inferred in Aristotle. (Part II, Chapter IV)
5. Much more detail on the nature of speculative and of practical knowledge is provided by Aquinas, and consequently on the speculative-practical character of both pure mathematics and the intermediate sciences. Except for one obscure reference which indicates otherwise, these sciences seem to be totally speculative for Aristotle. (Part II, Chapter IV)
6. St. Thomas does not agree with Aristotle that the sciences which apply mathematics to physical phenomena are simply mathematical sciences. Rather he refers to them as sciences which are neither just mathematical, nor just physical, but intermediate, between the two, sharing in both. (Part II, Chapter IV)

(The last three differences are the most important contributions St. Thomas makes to Aristotle's philosophy of mathe-

matics.)

7. Aquinas explicitly enunciates what I have called throughout this paper the <u>principle of dependent intelligibility</u> — viz. that one entity can be abstracted only from those entities upon whose intelligibility its intelligibility does not depend. Aristotle never enunciates this principle. And it is on the basis of this principle along with the statement of the ontological priority of quantity to other accidents (#2 above) that Thomas is able to explain in detail why mathematical abstraction involves no error (something Aristotle never does) and why the total abstracted mathematical object must contain substance as its substrate. This leads, then, to a more thorough and developed notion of intelligible matter as the substrate of the accident quantity and as necessarily present in the definition of mathematicals. On these last two points (the notion of intelligible matter as the necessary substrate of abstracted quantity and hence as necessarily present in the definition of the total abstracted mathematical object), the Stagirite, precisely because he does not express the principle of dependent intelligibility, is vague and incomplete. (Part II, Chapter II)

8. In general, Thomas has much more detail on the

nature of intelligible matter. He clearly sets forth a notion of common intelligible matter and he just as clearly identifies one sense of this with the substrate of the abstracted accident of quantity. Aristotle never makes such an identification nor does he ever speak of a common intelligible matter. As for the presence of this matter in the essence of the mathematical object, it is clear that for Aquinas common intelligible matter would be present in the essence of the total abstracted object, which is quantified substance. This is not clear in Aristotle. (Unfortunately, not even St. Thomas explicitly states whether common intelligible matter, referring now to the parts of mathematical quantities, is present in the essence of such quantities.) (Part II, Chapter II)

9. Finally, the very definite role given to imagination in mathematics by St. Thomas is solely his own contribution. That individual mathematicals as such are present only in imagination and, therefore, that mathematical judgments need terminate only there are exclusively Thomas's doctrines. They are at the most only obscurely hinted at in the Stagirite's teachings. (Part II, Chapter III)

There are two other points which though not expli-

citly stated by Aquinas, no more than they were by Aristotle, are nevertheless more easily inferred from his expressed teachings. I am referring to the freedom of the mathematician and the scope of the intermediate sciences.

1. The freedom of the mathematician: Through his emphasis on the role of imagination (#9 above), on the fact that individual mathematicals are imaginable, on the intellect as the proximate foundation for mathematical objects (#3 above), on construction as the means to show that mathematicals exist (#4 above) (which constructions must take place in imagination), and especially on the termination of mathematical judgments only in the imagination, St. Thomas has liberated the mathematician from a dependence on physical quantities much more than did Aristotle. The premises and conclusions of his science need only conform to imaginable entities. There is no intrinsic reason in Aquinas's doctrine why the mathematician must restrict himself to mathematical objects which resemble physical quantities.
2. The scope of the intermediate sciences: Because of his more detailed treatment of virtual quantity and of its ability to be measured and so translated into quantitative terms (#1 above), and especially because he is less optimistic than Aristotle as to the possibility of a physical

> science of nature in its specific features, St. Thomas paves the way for an increase in the role of the intermediate sciences in studying physical phenomena. These sciences are able to provide the detailed scientific knowledge unattainable by physics.

In conclusion, though the list of differences between him and Aristotle may be long, we must, of course, keep in mind that strictly speaking St. Thomas creates no philosophy of mathematics of his own. Even where he elaborates, where he disagrees, where he adds what Aristotle left out, he is always working within the context of the Stagirite's teaching on mathematics. The general areas of agreement between the two men on such basic issues as the object of mathematics, quantity, the nature of this accident, the doctrine of mathematical abstraction, the nature of abstracted quantity and its need for a substrate, attest to this. That Thomas had something to develop, to complete, is due to the Greek philosopher. For example, it is true that Aquinas is more complete about the nature of intelligible matter. Yet he is indebted to the Philosopher for originating such a notion in the first place. Indeed, most of the time it is on the basis of the Stagirite's own principles that St. Thomas develops and completes his doctrine. If the medieval doctor saw more about the nature of mathematics than the Greek philosopher — and he did (for the actual development in the aforementioned areas is his own achievement); one reason was because,

as he himself acknowledged, he was standing on the shoulders of a giant.

BIBLIOGRAPHY

Primary Sources

Aquinas, Thomas. Aristotle's De Anima in the Version of William of Moerbeke and the Commentary of St. Thomas Aquinas. Translated by Kenelm Foster, O.P., and Silvester Humphries, O.P. with an introduction by Ivo Thomas, O.P. London: Routledge and Kegan Paul. Ltd., 1931.

_____. Commentary on Aristotle's Physics. Translated by R. Blackwell, R. Spath, and W. E. Thirlkel. New Haven: Yale University Press, 1963.

_____. Commentary on Books I and II of the Physics in An Introduction to the Philosophy of Nature, compiled by R. A. Kocourek. St. Paul, Minn.: North Central Publishing Co., 1948.

_____. Commentary on the Metaphysics of Aristotle. Translated by John P. Rowan. 2 vols. "Library of Living Catholic Thought." Chicago: Henry Regnery Company, 1961.

_____. Commentary on the Nicomachean Ethics. Translated by C. I. Litzinger, O. P. 2 vols. Chicago: Henry Regnery Company, 1961.

_____. Exposition of the Posterior Analytics of Aristotle. Translated by Pierre Conway, O. P. Quebec, Canada: La Librairie Philosophique, M. Doyon, 1956.

_____. On Being and Essence. Translated by Armand Maurer. Toronto, Canada: Pontifical Institute of Mediaeval Studies, 1953.

_____. Opera Omnia. Parma Edition (1852-1873) with an introduction by Vernon J. Bourke. 25 vols. New York: Musurgia Publishers, 1949.

_____. Summa Theologica in Basic Writings of Saint Thomas Aquinas. Vol. I. Edited and annotated with an introduction by Anton C. Pegis. New York: Random House, 1945.

_____. The Division and Methods of the Sciences: Ques-

tions V and VI of His Commentary on the De Trinitate of Boethius. Translated with Introduction and Notes by Armand Maurer, O. S. B. Toronto, Canada: Pontifical Institute of Mediaeval Studies, 1953.

_____. The Principles of Nature in An Introduction to the Philosophy of Nature compiled by R. A. Kocourek. St. Paul, Minn.: North Central Publishing Co., 1948.

_____. The Summa Theologica of St. Thomas Aquinas. Literally translated by the Fathers of the English Dominican Province, 3rd ed. London: Burns, Oates and Washbourne, Ltd., 1938.

_____. Truth. Translated from the definitive Leonine text by Robert W. Mulligan, S. J. 3 vols. Chicago: Henry Regnery Company, 1952.

_____. On the Truth of the Catholic Faith. A translation of the Summa Contra Gentiles by A. C. Pegis, J. P. Anderson, V. J. Bourke, and Charles O'Neil. 5 vols. New York: Doubleday Image Books, 1954-1956.

Aristotle. Aristotle's Metaphysics. Translated by Richard Hope. New York: Columbia University Press, 1952.

_____. Aristotle's Physics. Translated by Richard Hope with an Analytical Index of Technical Terms. Lincoln, Nebraska: University of Nebraska Press, 1961.

_____. De Anima. Translated with introduction and notes by R. D. Hicks. Cambridge: University Press, 1907.

_____. The Basic Works of Aristotle. Edited with an introduction by Richard McKeon. New York: Random House, 1941.

_____. The Works of Aristotle. Edited by W. D. Ross. 12 vols. London: Oxford University Press, 1950-1956.

Secondary Sources

Books

Anderson, James. The Bond of Being. St. Louis: Herder Book Co., 1949.

Apostle, Hippocrates George. Aristotle's Philosophy of Mathematics. Chicago: University of Chicago Press, 1952.

Bennett, Owen. The Nature of Demonstrative Proof — ac-

cording to the Principles of Aristotle and St. Thomas Aquinas. Washington, D. C.: Catholic University Press, 1943.

Bobik, Joseph. Saint Thomas on the Individuation of Bodily Substances. Unpublished doctoral dissertation, Notre Dame, 1953.

Bourke, Vernon. Aquinas' Search for Wisdom. Milwaukee: Bruce Publishing Co., 1965.

Cajori, F. A History of Mathematics. New York: The Macmillan Company, 1951.

Clagett, Marshall. Greek Science in Antiquity. New York: Abelard-Schuman, Inc., 1955.

Cohen, M. R. and Drabkin, I. E. A Source Book in Greek Science. Cambridge: Harvard University Press, 1958.

Collingwood, Francis J. Man's Physical and Spiritual Nature. New York: Holt, Rinehart and Winston, Inc., 1963.

_____. Philosophy of Nature. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1961.

Crombie, A. C. Medieval and Early Modern Science. 2 vols. Garden City, N.Y.: Doubleday Anchor Books, 1959.

_____. Robert Grossteste and the Origins of Experimental Science. 1100-1700. Oxford: Clarendon Press, 1953.

Defarrari, Roy J., Barry, Sister M. Inviolata, O. D. P., and Mc Guiness, Ignatius, O. P. A Lexicon of St. Thomas Aquinas. Washington, D. C.: Catholic University Press (John D. Lucas Printing Co., Baltimore), 1948.

De Bergomo, Petri. In Opera Sancti Thomae Aquinatis Index: Seu Tabula Aurea Eximii Doctoris. Editiones Paulinae. Editio Fototypica. New York: n.d.

Duhem, Pierre. Le système du monde. 7 vols. Paris: Librairie Scientifique, A. Hermann et Fils, 1913.

Eves, H. An Introduction to the History of Mathematics. New York: Holt, Rinehart and Winston, Inc., 1961.

Gilson, Etienne. The Christian Philosophy of St. Thomas Aquinas. Translated by L. K. Shook, C. S. B. with A Catalogue of St. Thomas's Works by I. T. Eschmann, O. P. New York: Random House, 1956.

Greenwood, Thomas. Études sur La Connaissance Mathématique. Ottawa: Ottawa University Press, 1942.

Heath, Thomas. A History of Greek Mathematics. 2 vols. Oxford: Clarendon Press, 1960.

_____. Mathematics in Aristotle. Oxford: Clarendon Press, 1949.

Jaeger, Werner. Aristotle: Fundamentals of the History of the Development of His Thought. Translated by Richard Robinson. 2d ed. Oxford: University Press, 1948.

Mahoney, Bertrand. The Notion of Quantity in a Thomistic Evaluation of the Sciences. River Forest, Ill., 1951.

Mansion, A. Introduction à la Physique Aristotélicienne. 2d ed. Louvain: Instit. Super. de Philosophie, 1946.

Maritain, Jacques. Philosophy of Nature. New York: Philosophical Library, 1951.

_____. The Degrees of Knowledge. Translated from the French under the supervision of Gerald B. Phelan. 4th ed. New York: Charles Scribner's Sons, 1959.

Maziarz, Edward. The Philosophy of Mathematics. New York: Philosophical Library, 1950.

Morrison, John. A Definition of Mathematics. Unpublished doctoral dissertation, Notre Dame, 1951.

Neugebauer, Otto. The Exact Sciences in Antiquity. Princeton, N.J.: Princeton University Press, 1952.

Organ, Troy Wilson. An Index to Aristotle in English Translation. Princeton, N.J.: Princeton University Press, 1949.

Owens, Joseph. A History of Ancient Western Philosophy. New York: Appleton-Century-Crofts, Inc., 1959.

_____. An Elementary Christian Metaphysics. Milwaukee: Bruce Publishing Co., 1963.

_____. The Doctrine of Being in the Aristotelian Metaphysics. 2d printing, revised. Toronto, Canada: Pontifical Institute of Mediaeval Studies, 1957.

Ross, W. D. Aristotle. London: Methuen and Co., Ltd., 1956.

_____. Aristotle's Metaphysics. A revised Text with Introduction and Commentary. 2 vols. Oxford: University Press, 1958.

Schutz, Ludwig. Thomas-Lexikon. New York: Musurgia Publishing Co., n.d. (First published in 1895: Pader-

born, Druck und Verlag von Ferdinand Schoningh)

Simmons, Edward D. The Scientific Art of Logic. Milwaukee: Bruce Publishing Co., 1961.

_____. The Thomistic Doctrine of Intellectual Abstraction for the Three Levels of Science: Exposition and Defense. Ann Arbor, Mich.: University Microfilms, 1954.

Smith, Gerard, S. J. and Kendzierski, Lottie H. The Philosophy of Being. New York: The Macmillan Company, 1961.

Smith, Vincent Edward. St. Thomas on the Object of Geometry. Milwaukee: Marquette University Press, 1954.

_____. The General Science of Nature. Milwaukee: Bruce Publishing Co., 1958.

Struik, D. A Concise History of Mathematics. New York: Dover Publishing Co., 1948.

Taylor, A. E. Aristotle. New York: Dodge, n.d.

_____. Aristotle on His Predecessors. Translated from the text edition of W. Christ with introduction and notes. 2d ed. Chicago: The Open Court Publishing Co., 1910.

van der Waerden, B. L. Science Awakening. New York: Science Editions, 1964.

Zeller, Eduard. Aristotle and the Earlier Peripatetics. Vol. IV, 1 and 2, of Philosophy of the Greeks. Translated by B. F. C. Costelloe and J. H. Muirhead. New York: Longmans, Green and Co., 1897.

Articles

Alvarez, Giuseppe. "La filosofìa della matematica in San Tommaso," Revista di Filosofìa neo-Scholastica, XLII (1950), 142-53.

Augustin-Gabriel, Frère. "Matière intelligible et mathématique," Laval Théologique et Philosophique, XVII (1961), 173-96.

_____. "Matière intelligible et mathématique, II," Laval Théologique et Philosophique, XVIII (1962), 177-210.

Brennan, R. "The Thomistic Concept of Imagination," New Scholasticism, XV (1941), 149-61.

Cantin, S. "L'âme et ses puissances selon Aristote," Laval Théologique et Philosophique, II (1946), 184-205.

_____. "L'intelligence selon Aristote," Laval Théologique et Philosophique, IV (1948), 252-88.

_____. "La mémoire et la reminiscence d'après Aristote," Laval Théologique et Philosophique, XI (1955), 81-89.

Collingwood, Francis J. "Intelligible Matter in Contemporary Science," Proceedings of the American Catholic Philosophical Association, XXXVIII (1964), 109-18.

_____. "Is 'Physical Knowledge' Limited by Its Quantitative Approach to Reality?" in The Nature of Physical Knowledge, edited by L. W. Friedrich, S.J. (London: Oliver and Boyd, 1961), pp. 25-46.

De Koninck, Charles. "Abstraction from Matter: Notes on St. Thomas's Prologue to the Physics," Laval Théologique et Philosophique, XIII (1957), 133-96.

_____. "Abstraction from Matter, II," Laval Théologique et Philosophique, XVI (1960), 53-69.

_____. "Preface" to Précis de Psychologie Thomiste by Stanislas Cantin. (Quebec, Canada: Laval University Press, 1948), pp. vii-lxxxiii.

Gehring, R. "The knowledge of material essences according to St. Thomas Aquinas," Modern Schoolman, XXXIII (1956), 153-81.

Geiger, L. B. "Abstraction et separation d'après S. Thomas," Revue Des Sciences Philosophiques et Théologiques, XXI (1947), 3-40.

Gerhard, W. "Natural Sciences and the Imagination," Thomist, XVI (1953), 190-216.

Greenwood, Thomas. "Aristotle on Mathematical Constructibility," Thomist, XVII (1954), 84-94.

_____. "La nature de mathématiques selon Aristote," Rev. Univ. Ottawa, XXII (1953), 216-28.

_____. "The Characters of the Aristotelian Logic," Thomist, IV (1942), 221-46.

Heath, Mark. "Can Qualities Be Measured?" Thomist, XVIII (Jan. '55), 31-60.

Kane, W. "Abstraction and the distinction of the sciences," Thomist, XVII (1954), 43-68.

Leroy, M-V. "Le savoir spéculatif," and "Annexe: Abstractio et separatio d'après un texte controverse de saint

Thomas," Revue Thomiste, XLVIII (1948), 236-339.

Lonergan, Bernard. "Note on geometrical possibility," Modern Schoolman, XXVII (1950), 124-38.

Mansion, A. "La physique aristotélicienne et la philosophie," Revue Néoscholastique de Philosophie, XXXIX (1936), 5-26.

Maurer, Armand. "A Neglected Thomistic Text on the Foundation of Mathematics," Medieval Studies, XXI (1959), 185-92.

_____. "Form and Essence in St. Thomas," Medieval Studies, XIII (1951), 165-76.

Mullahy, B. "Subalternation and Mathematical Physics," Laval Théologique et Philosophique, II (1946), 89-101.

_____. "The Nature of the Liberal Arts," New Scholasticism, XXIII (1949), 361-86.

Mullaney, James V. "The Liberal Arts in the Aristotelian Thomist Scheme of Knowledge," Thomist, XIX (1956), 481-96.

Nemetz, A. "Logic and the Division of the Sciences in Aristotle and Thomas Aquinas," Modern Schoolman, XXXIII (1956), 91-109.

O'Donnell, Robert A. "Individuation: an Example of the Development in the Thought of St. Thomas Aquinas," New Scholasticism, XXXIII (1959), 49-67.

Oesterle, J. "Theoretical and Practical Knowledge," Thomist, XXI (1958), 146-61.

Owens, Joseph. "Our Knowledge of Nature," Proceedings of the American Catholic Philosophical Association, XXIX (1955), 63-86.

_____. "The Aristotelian Conception of the Sciences." International Philosophical Quarterly, IV (1964), no. 2, 200-16.

Philippe, M.D. "Abstraction, Addition, Séparation dans la Philosophie Aristote," Revue Thomiste, XLVIII (1948), 461-79.

Pro, Diego. "Filosofía de la matemática en Aristóteles," Sapientia, XI (1956), 78-99.

Roth, Robert. "The Aristotelian Use of Phantasia and Phantasma," New Scholasticism, XXXVII (July, 1963),

312-26.

Saint-Éduoard, Mère, O. S. O. "La division aristotélicienne des sciences, selon le professeur A. Mansion," Laval Théologique et Philosophique, XV (1959), 215-35.

Simon, Yves R. "Maritain's Philosophy of the Sciences" in Philosophy of Nature by Jacques Maritain (New York: Philosophical Library, 1951), pp. 157-82.

_____. "Nature and the Process of Mathematical Abstraction," Thomist, XXIX (April, 1965), 117-39.

Smith, Robert. "The Liberal Arts," Laval Théologique et Philosophique, II (1946), 79-88.

Whittaker, John F. "Mathematics in Speculative Science," Thomist, III (1941), 467-506.

Winance, Éleuthère, O. S. B. "Note sur l'abstraction mathématique selon saint Thomas," Revue Philosophique de Louvain, LIII (1955), 482-510.

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